THEOREM 3.2. *A mapping f of a set $\mathfrak{K} \subset \mathfrak{A}$ in an $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{B}$ can be extended to an $\mathfrak{m}$-homomorphism h of $\mathfrak{K}_{\mathfrak{m}}$ in $\mathfrak{B}$ if and only if for every set $\{A_t: t \in T, \overline{\overline{T}} \leqslant \mathfrak{m}\} \subset \mathfrak{K}$*

(i) $\bigcap_{t\in T} \varepsilon(t) A_t = \wedge$ *implies* $\bigcap_{t\in T} \varepsilon(t) f(A_t) = \wedge$,

where $\varepsilon(t) = 1$ *or* -1 *for every* $t \in T$.

Proof. The necessity is obvious. By lemma 3.1 we must prove the sufficiency of (i) only in the case where the power of $\mathfrak{K}$ is $\leqslant \mathfrak{m}$.

In this case, however, the power of $f(\mathfrak{K})$ is also $\leqslant \mathfrak{m}$. Hence $f(\mathfrak{K})_{\mathfrak{m}}$ is isomorphic with an $\mathfrak{m}$-complete field of sets, by $\mathfrak{m}$-distributivity of $\mathfrak{B}$.

Therefore, by (B), the mapping f of $\mathfrak{K}$ into $f(\mathfrak{K})_{\mathfrak{m}}$ can be extended to an $\mathfrak{m}$-homomorphism

$$h_{\mathfrak{K}}: \mathfrak{K}_{\mathfrak{m}} \to f(\mathfrak{K})_{\mathfrak{m}},$$

i. e. to an $\mathfrak{m}$-homomorphism $h_{\mathfrak{K}}: \mathfrak{K}_{\mathfrak{m}} \to \mathfrak{B}$, q. e. d.

4. The proof of theorem 1.1. Let $\{\mathfrak{A}_t\}_{t\in T}$ be an indexed set of non-degenerate $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebras. Let $\mathfrak{B}$ be the minimal $\mathfrak{m}$-product of these algebras. By 2.3, $\mathfrak{B}$ is $\mathfrak{m}$-distributive.

Let $\mathfrak{C}$ be any $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra. By the definition of free $\mathfrak{m}$-distributive product of an indexed set of Boolean algebras (see the introduction) it remains to prove that if, for every $t \in T$, h_t is an $\mathfrak{m}$-homomorphism of $i_t(\mathfrak{A}_t)$ into $\mathfrak{C}$, then there exists an $\mathfrak{m}$-homomorphism h of $\mathfrak{B}$ into $\mathfrak{C}$ which is a common extension of all the homomorphisms h_t.

This follows, however, immediately from 3.2. Condition (i) is satisfied since the subalgebras $i_t(\mathfrak{A}_t)$ of $\mathfrak{B}$ are $\mathfrak{m}$-independent.

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Reçu par la Rédaction le 18. 12. 1962

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

MINIMAL EXTENSIONS OF WEAKLY DISTRIBUTIVE BOOLEAN ALGEBRAS

BY

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Introduction. Pierce [2] has proved two important theorems on minimal extensions of $\mathfrak{m}$-distributive Boolean algebras. The purpose of the present paper is to generalize those theorems to weakly $\mathfrak{m}$-distributive Boolean algebras.

Terminology and notation. The symbol $\bigcup$ will be used both for the Boolean join and for the set-theoretical union. The symbol $\bigcap$, similarly, will be used both for the Boolean meet and for the set-theoretical intersection. The zero element of a Boolean algebra will be denoted by 0 and the unit element by 1.

A Boolean algebra and the set of all its elements will be denoted by the same letter.

A subset A of a Boolean algebra B is said to be a *covering* of B if $\bigcup_{a\in A} a = 1$.

A covering A of a Boolean algebra B is said to be $\mathfrak{m}$-*covering* of B if $\overline{\overline{A}} \leqslant \mathfrak{m}$, where $\overline{\overline{A}}$ denotes the cardinal number of A. A covering or $\mathfrak{m}$-covering A is called *partition*, respectively $\mathfrak{m}$-*partition* if elements of A are disjoint.

If A and C are subsets of a Boolean algebra B, we say that A *refines* C, if for every $a \in A$ there exists $c \in C$ such that $a \subset c$; we say that A *weakly refines* C if for every $a \in A$ there exists a finite sequence

$$(c_1, c_2, \ldots, c_k) \subset C$$

such that $a \subset \bigcup_{i=1}^{k} c_i$.

A subalgebra B_2 of a Boolean algebra B_1 is said to be an $\mathfrak{m}$-*regular subalgebra* of B_1, when for every set $A \subset B_2$, $\overline{\overline{A}} \leqslant \mathfrak{m}$, if the join $\bigcup_{a\in A} a$ exists in B_2, it is also the join of this set in B_1. If B_2 is an $\mathfrak{m}$-regular subalgebra of B_1 for every infinite cardinal $\mathfrak{m}$, then B_2 is said to be a *regular subalgebra* of B_1.

If $\mathfrak{m}$ is a cardinal number, then $\mathfrak{m}^+$ will denote the successor, i. e. the least cardinal number $> \mathfrak{m}$.

If B is a Boolean algebra, then $B^{\mathfrak{m}}$ will denote the minimal $\mathfrak{m}$-extension of B, i. e. $B^{\mathfrak{m}}$ is an $\mathfrak{m}$-complete Boolean algebra, B is dense in $B^{\mathfrak{m}}$ and $\mathfrak{m}$-generates $B^{\mathfrak{m}}$.

By *minimal extension* of B we mean a complete Boolean algebra $\mathfrak{B}^\infty$ which contains B as a dense subalgebra.

If S and T are non-empty sets, then the set of all mappings of T into S will be denoted by S^T, as usually.

1. Weak $(\mathfrak{m}, \mathfrak{n})$**-distributivity.** We have

1.1. Definition. A Boolean algebra B is said to be *weakly* $(\mathfrak{m}, \mathfrak{n})$-*-distributive* if

$$\bigcap_{t\in T} \bigcup_{s\in S} a_{t,s} = \bigcup_{\Phi\in \mathbf{S}^T} \bigcap_{t\in T} a_{t,\Phi(t)},$$

where $\mathbf{S}$ is the class of all finite subsets of S, $\overline{\overline{T}} \leqslant \mathfrak{m}$, $\overline{\overline{S}} \leqslant \mathfrak{n}$, $a_{t,\Phi(t)} = \bigcup_{s\in\Phi(t)} a_{t,s}$ and as well the join $\bigcup_{s\in S} a_{t,s}$ as the meets $\bigcap_{t\in T}\bigcup_{s\in S} a_{t,s}$, $\bigcap_{t\in T} a_{t,\Phi(t)}$ exist in B (see, e. g. [3], p. 102).

We note some immediate consequences of 1.1.

1.2. *If a Boolean algebra B is weakly* $(\mathfrak{m}, \mathfrak{n})$*-distributive and* $\mathfrak{m}' < \mathfrak{m}$, $\mathfrak{n}' < \mathfrak{n}$, *then B is weakly* $(\mathfrak{m}', \mathfrak{n}')$*-distributive.*

1.3. *A regular subalgebra of a weakly* $(\mathfrak{m}, \mathfrak{n})$*-distributive Boolean algebra is also weakly* $(\mathfrak{m}, \mathfrak{n})$*-distributive.*

1.4. *Every Boolean algebra is weakly* $(k, \mathfrak{n})$*-distributive and weakly* $(\mathfrak{n}, k)$*-distributive, where k is a finite integer and $\mathfrak{n}$ is an arbitrary cardinal number.*

In the sequel we suppose that $\mathfrak{m}$ and $\mathfrak{n}$ are infinite cardinals.

Now the following criterion for a Boolean algebra to be weakly $(\mathfrak{m}, \mathfrak{n})$-distributive is presented.

1.5. Theorem. *A Boolean algebra B is weakly* $(\mathfrak{m}, \mathfrak{n})$*-distributive if and only if for every class*

$$\{A_t : t\in T,\ \overline{\overline{T}} \leqslant \mathfrak{m}\}$$

of $\mathfrak{n}$-coverings of B there exists a covering A of B which weakly refines every A_t.

Proof of necessity. Let

$$A_t = \{a_{t,s} : s\in S,\ \overline{\overline{S}} \leqslant \mathfrak{n}\}$$

and let

$$A = \{a\in B : \{a\} \text{ weakly refines every } A_t\}.$$

Obviously A weakly refines every A_t. Suppose that A is not a covering of B. Then there is some $b \neq 0$, $b\in B$, disjoint with every $a\in A$.

By the weakly $(\mathfrak{m}, \mathfrak{n})$-distributivity of B there exists (see [3], p. 103) a mapping $\Phi \in \mathbf{S}^T$ such that

$$b \cap \bigcap_{t\in T} a_{t,\Phi(t)} \neq 0.$$

This leads to a contradiction because the meet $\bigcap_{t\in T} a_{t,\Phi(t)}$ belongs to A.

Proof of sufficiency. Let

$$\{b_{t,s} : t\in T,\ s\in S,\ \overline{\overline{T}} \leqslant \mathfrak{m},\ \overline{\overline{S}} \leqslant \mathfrak{n}\} \subset B.$$

We suppose now the existence of

$$\bigcup_{s\in S} b_{t,s} \text{ for every } t\in T, \qquad \bigcap_{t\in T}\bigcup_{s\in S} b_{t,s} = b,$$

and

$$\bigcap_{t\in T} b_{t,\Phi(t)} \text{ for every } \Phi\in \mathbf{S}^T.$$

Let

$$s_0 \notin S, \qquad S_0 = S \cup \{s_0\}, \qquad b_{t,s_0} = b'$$

for every $t\in T$ and let

$$A_t = \{b_{t,s} : s\in S_0\}.$$

In this way every A_t becomes a covering of B and then, by the assumption, there exists a covering A which weakly refines every A_t. Therefore there exists a mapping $\Phi\in \mathbf{S}_0^T$ such that

$$\bigcap_{t\in T} b_{t,\Phi(t)} \neq 0,$$

and this means that the algebra B is weakly $(\mathfrak{m}, \mathfrak{n})$-distributive (see [3], p. 103).

The following two statements will be useful in the sequel.

1.6. *If a Boolean algebra B satisfies the $\mathfrak{n}$-chain condition (i. e. every partition is an $\mathfrak{n}$-partition), then the following conditions are equivalent:*

(i) *B is weakly* $(\mathfrak{m}, \mathfrak{n})$*-distributive;*

(ii) *for every family* $\{A_t : t\in T, \overline{\overline{T}} \leqslant \mathfrak{m}\}$ *of $\mathfrak{n}$-partitions of B there exists a covering A of B which weakly refines every A_t.*

1.7. *If a Boolean algebra B is $\mathfrak{n}'$-complete for every $\mathfrak{n}' < \mathfrak{n}$, then conditions* (i) *and* (ii) *from* 1.6 *are also equivalent.*

Proof of 1.6. Obviously (i) $\Rightarrow$ (ii). The proof of (ii) $\Rightarrow$ (i) is based on the well-ordering axiom.

Let A be any $\mathfrak{n}$-covering of B. Let

$$a_1, a_2, \ldots, a_a, \ldots \qquad (a < \beta)$$

be a transfinite sequence of all elements of A, where β is the least ordinal number of power $\mathfrak{n}$. We denote by B^∞ the minimal extension of B. Let us define in B^∞:

$$(1) \qquad b_1 = a_1,\ b_2 = a_2 \cap a_1',\ \ldots,\ b_a = a_a \cap \bigcap_{\gamma < a} a_\gamma' \ \ldots$$

(a' is the complement of a). The set

$$A = \{b_a : a < \beta\}$$

is an $\mathfrak{n}$-partition of B^∞ such that $b_a \subset a_a$ for every $a < \beta$.

B being a dense subset of B^∞, every b_a is a join of disjoint elements of B. By the $\mathfrak{n}$-chain condition the set of all these elements has a cardinal number $\leqslant \mathfrak{n}$. Thus there exists in B an $\mathfrak{n}$-partition which refines A.

Now let

$$\{A_t : t \in T,\ \overline{\overline{T}} \leqslant \mathfrak{m}\}$$

be a family of $\mathfrak{n}$-coverings of B. Let C_t be an $\mathfrak{n}$-partition which refines A_t. In view of (ii) there exists a covering A which weakly refines every C_t; evidently, it weakly refines every A_t, too, q. e. d.

Proof of 1.7. The Boolean algebra B being $\mathfrak{n}'$-complete for every $\mathfrak{n}' < \mathfrak{n}$, the formulas (1) define an $\mathfrak{n}$-partition of B which refines A. The remaining part of the proof is the same as in 1.6.

1.8. Definition. A Boolean algebra is said to be *weakly* $\mathfrak{m}$-*distributive* if it is weakly $(\mathfrak{m}, \mathfrak{m})$-distributive.

1.9. THEOREM. *A minimal extension of a weakly* $\mathfrak{m}$-*distributive Boolean algebra* B *satisfying the* $\mathfrak{m}$-*chain condition is also weakly* $\mathfrak{m}$-*distributive.*

Proof. Let B^∞ be the minimal extension of B. Since B is dense in B^∞ and the $\mathfrak{m}$-chain condition is fulfilled, it follows that for every family

$$\{\overline{A}_t : t \in T,\ \overline{\overline{T}} \leqslant \mathfrak{m}\}$$

of $\mathfrak{m}$-partitions of B^∞ there exists a family $\{A_t : t \in T\}$ of $\mathfrak{m}$-partitions of B such that every A_t refines $\overline{A}_t$.

Therefore if a covering of B weakly refines every A_t, then it weakly refines every $\overline{A}_t$, too. Thus B^∞ is weakly $\mathfrak{m}$-distributive, by 1.7.

It may be asked whether the $\mathfrak{m}$-chain condition is necessary in 1.9 (**P 433**). I am not able to give an answer.

2. Weak $\mathfrak{m}$-distributivity of minimal extensions of fields of sets.

2.1. Definition. If A and C are subsets of a Boolean algebra B, then A is said to $\mathfrak{m}$-*refine* C if and only if for every $a \in A$ there exists a subset of C

$$\{a_t : t \in T,\ \overline{\overline{T}} \leqslant \mathfrak{m}\} \subset C$$

such that $a \subset \bigcup_{t \in T} a_t$.

In this section we suppose that a denotes the least ordinal number of power $\mathfrak{m}$.

2.2. LEMMA. *If an* $\mathfrak{m}$-*complete Boolean algebra* B *is weakly* $\mathfrak{m}$-*distributive and for every* $\mathfrak{n}$-*partition* A *of* B *there exists a transfinite sequence* $\{A_\xi : \xi < a\}$ *of* $\mathfrak{m}$-*coverings of* B

$$A_\xi = \{a_{\xi,\eta} : \eta < a\} \quad \textit{for every} \quad \xi < a$$

such that

(1) *every* $a_{\xi,\eta}$ *is a join of some elements of* A,

(2) *for every transfinite sequence of finite sequences* $\eta_1(\xi), \eta_2(\xi), \ldots, \eta_{k_\xi}(\xi)$ *of ordinal numbers* $< a$ *the meet*

$$\bigcap_{\xi < a} (a_{\xi,\eta_1(\xi)} \cup a_{\xi,\eta_2(\xi)} \cup \ldots \cup a_{\xi,\eta_{k_\xi}(\xi)})$$

is a join of at most $\mathfrak{m}$ *elements of* A,

then for every family

$$\{A_t : t \in T,\ \overline{\overline{T}} \leqslant \mathfrak{m}\}$$

of $\mathfrak{n}$-*partitions of* B *there exists a covering* C *which* $\mathfrak{m}$-*refines every* A_t.

Proof. For every $t \in T$ let $\{A_\xi^t : \xi < a\}$ be a sequence of $\mathfrak{m}$-coverings of B,

$$A_\xi^t = \{a_{\xi,\eta}^t : \eta < a\}$$

such that

(1′) every $a_{\xi,\eta}^t$ is the join of some elements of A_t,

(2′) for every finite sequence $\eta_1^t(\xi), \eta_2^t(\xi), \ldots, \eta_{k_\xi}^t(\xi)$ of ordinal numbers $< a$ the meet

$$\bigcap_{\xi < a} (a_{\xi,\eta_1(\xi)}^t \cup a_{\xi,\eta_2(\xi)}^t \cup \ldots \cup a_{\xi,\eta_{k_\xi}(\xi)}^t)$$

is a join of at most $\mathfrak{m}$ elements of A_t.

Since the Boolean algebra B is weakly $\mathfrak{m}$-distributive and $\mathfrak{m}^2 = \mathfrak{m}$, it follows that there exists a covering C of B which weakly refines every A_ξ^t, i. e. for every $c \in C$, and for every $t \in T$, and every $\xi < \alpha$, there exists a finite sequence $\eta_1^t(\xi), \eta_2^t(\xi), \ldots, \eta_{k_\xi}^t(\xi)$ of ordinal numbers $< \alpha$ such that

$$c \subset \bigcap_{\xi<\alpha} (a^t_{\xi,\eta_1(\xi)} \cup a^t_{\xi,\eta_2(\xi)} \cup \ldots \cup a^t_{\xi,\eta_{k_\xi}(\xi)}) \quad \text{for every} \quad t \in T.$$

It follows from (2') that C $\mathfrak{m}$-refines every A_t, and this completes the proof.

The following lemma belongs to the General Theory of Sets:

2.3. LEMMA. *If $\overline{\overline{A}} = \mathfrak{m}^+ = 2^{\mathfrak{m}}$, then there exists a double sequence $\{a_{\xi,\eta}: \xi < \alpha, \eta < \alpha\}$ of subsets of A such that*

$$\bigcup_{\eta<\alpha} a_{\xi,\eta} = A \quad \textit{for every} \quad \xi < \alpha,$$

and condition (2) *of lemma* 2.2 *is satisfied.*

This lemma was proved by S. Banach and C. Kuratowski [1] for $\mathfrak{m} = \aleph_0$. For $\mathfrak{m} > \aleph_0$ only slight modifications are necessary.

Proof. Let us consider the set F of mappings of the set of ordinal numbers $< \alpha$ into itself. For $\varphi, \psi \in F$ we say

$$\varphi \leqslant \psi \text{ if and only if } \varphi(\xi) \leqslant \psi(\xi) \text{ for every } \xi < \alpha.$$

For every subset $\Phi \subset F$ of power $\mathfrak{m}$ we can define (by diagonal method) a mapping $\psi \in F$ such that

$$\psi \leqslant \varphi \text{ does not hold for every } \varphi \in \Phi.$$

By the assumption, $\overline{\overline{F}} = \mathfrak{m}^{\mathfrak{m}} = 2^{\mathfrak{m}} = \mathfrak{m}^+$. Let

$$F = \{\varphi_\xi: \xi < \beta\}$$

be a transfinite sequence of all elements of F, where β is the least ordinal number of power $\mathfrak{m}^+$. Then for every $\eta < \beta$ there exists $\varphi_{\gamma_\eta} \in F$ such that

$$\varphi_{\gamma_\eta} \leqslant \varphi_\xi \text{ does not hold for every } \xi \leqslant \eta.$$

We may suppose that $\eta \neq \eta'$ implies $\varphi_{\gamma_\eta} \neq \varphi_{\gamma_{\eta'}}$. Let Φ_0 be a set of all such mappings φ_{γ_η} $(\eta < \beta)$. Of course, the set of all mappings $\psi \in \Phi_0$, for which $\psi \leqslant \varphi_\xi$, has a cardinal number $\leqslant \mathfrak{m}$.

Since $\overline{\overline{A}} = \overline{\overline{\Phi}}_0 = \mathfrak{m}^+$, we may index the set A by elements of Φ_0:

$$A = \{a_\varphi: \varphi \in \Phi_0\}.$$

Let $a_{\xi,\eta}$ be the join of all a_φ for which $\varphi(\xi) = \eta$.

Evidently

$$\bigcup_{\eta<\alpha} a_{\xi,\eta} = 1 \quad \text{for every} \quad \xi < \alpha.$$

For every $\xi < \alpha$ let us construct a finite sequence $\eta_1(\xi), \eta_2(\xi), \ldots, \eta_{k_\xi}(\xi)$ of ordinals $< \alpha$. We define

$$\varphi_0(\xi) = \max(\eta_1(\xi), \eta_2(\xi), \ldots, \eta_{k_\xi}(\xi)).$$

Obviously $\varphi_0 \in F$. Therefore, if

(i) $a_\varphi \subset \bigcap_{\xi<\alpha} (a_{\xi,\eta_1(\xi)} \cup a_{\xi,\eta_2(\xi)} \cup \ldots \cup a_{\xi,\eta_{k_\xi}(\xi)})$,

then $\varphi \leqslant \varphi_0$, and $\varphi \in \Phi_0$.

Consequently the set of all elements a_φ satisfying (i) is of power $\leqslant \mathfrak{m}$, q. e. d.

2.4. LEMMA. *If an $\mathfrak{m}^+$-complete Boolean algebra B is weakly $\mathfrak{m}$-distributive and $\mathfrak{m}^+ = 2^{\mathfrak{m}}$, then for every class*

$$\{A_t: t \in T, \overline{\overline{T}} \leqslant \mathfrak{m}\}$$

of $\mathfrak{m}^+$-partitions of B there exists a covering C which $\mathfrak{m}$-refines every A_t.

Proof. By 2.3, the assumptions of 2.2 are satisfied where $\mathfrak{n} = \mathfrak{m}^+$, and this completes the proof.

2.5. LEMMA. *If the minimal $\mathfrak{n}$-extension $B^{\mathfrak{n}}$ of a Boolean algebra B is weakly $\mathfrak{m}$-distributive and if we suppose*

$$\mathfrak{n} \geqslant \mathfrak{m}^+ = 2^{\mathfrak{m}},$$

then for every family $\{A_t: t \in T, \overline{\overline{T}} \leqslant \mathfrak{m}\}$ of $\mathfrak{m}^+$-coverings of B there exists a covering C of B which $\mathfrak{m}$-refines every A_t.

Proof. The Boolean algebra $B^{\mathfrak{n}}$ satisfies all the assumptions of 2.4. Since $B^{\mathfrak{n}}$ is $\mathfrak{m}^+$-complete, it follows that every $\mathfrak{m}^+$-covering of $B^{\mathfrak{n}}$ is refined by an $\mathfrak{m}^+$-partition of $B^{\mathfrak{n}}$ (see [2], and the proof of 1.7). Obviously every covering of B is a covering of $B^{\mathfrak{n}}$.

Consequently, by 2.4, for every family $\{A_t: t \in T, \overline{\overline{T}} \leqslant \mathfrak{m}\}$ of $\mathfrak{m}^+$-coverings of B there exists in $B^{\mathfrak{n}}$ a covering $\overline{C}$ which $\mathfrak{m}$-refines every A_t.

Since B is a dense subalgebra of $B^{\mathfrak{n}}$, there exists in it a covering C which refines $\overline{C}$, and thus $\mathfrak{m}$-refines every A_t. This completes the proof of the lemma.

2.6. THEOREM. *If $\mathfrak{m}^+ = 2^{\mathfrak{m}}$, then there exists an $\mathfrak{m}$-complete field of sets F such that its minimal extension is not weakly $\mathfrak{m}$-distributive.*

Proof. Pierce [2] has built an $\mathfrak{m}$-complete field of sets F whose minimal extension is not $\mathfrak{m}$-distributive.

Namely: let X be an arbitrary set of power $\mathfrak{m}^+$, let Y be the set of all ordinal numbers smaller than α, where α is the least ordinal number of power $\mathfrak{m}$, and let Z be a set of mappings of X into Y, defined as follows:

$$f \in Z \text{ if and only if } f(x) < \eta \text{ for every } x \in X, \text{ and some } \eta \in Y.$$

For every subset $W \subset X$, $\overline{\overline{W}} \leqslant \mathfrak{m}$, and for every $\varphi \in Z$, let $L_{W,\varphi}$ be a subset of Z defined as follows:

$$f \in L_{W,\varphi} \text{ if and only if } f|W = \varphi|W$$

(i. e. if $f(x) = \varphi(x)$ for every $x \in W$).

F is the $\mathfrak{m}$-field of subsets of Z generated by all the sets $L_{W,\varphi}$.

Now we are going to prove that $F^{\mathfrak{m}}$ is not weakly $\mathfrak{m}$-distributive, if $\mathfrak{n} \geqslant \mathfrak{m}^+$.

Let

$$T(x, \eta) = \{f \in Z : f(x) = \eta\},$$

and let

$$A_\eta = \{T(x, \eta) : x \in X\}.$$

Every family A_η, $\eta \in Y$, $\mathfrak{m}^+$-covers F (see [2], p. 139).

It follows from the definition of Z that

(i) $$\bigcap_{\eta \in Y} \bigcup_{x \in X} T(x, \eta) = 0,$$

where the intersection and the union are set-theoretical.

Suppose that there exists a covering A of F which $\mathfrak{m}$-refines every A_η. Thus, the field F being $\mathfrak{m}$-complete, every element of A is included in the set-theoretical union of elements of A_η, for every $\eta \in Y$. Therefore it is empty, by (i). Contradiction.

Consequently, by lemma 2.5, $F^{\mathfrak{m}}$ is not weakly $\mathfrak{m}$-distributive.

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Reçu par la Rédaction le 8. 8. 1962

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

A FEW PROBLEMS ON BOOLEAN ALGEBRAS

BY

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The purpose of this short note is to collect a few problems concerning Boolean algebras which seem to be interesting. Some of them were mentioned in my expository paper [7], others were quoted in my book [9]. Perhaps, the level of difficulty of some of them is rather low. In any case, their solutions will mean a progress in the theory of Boolean algebras.

The first problem concerns the following simple theorem: If a Boolean algebra $\mathfrak{A}$ is $\mathfrak{m}'$-complete for every infinite cardinal $\mathfrak{m}' < \mathfrak{m}$, $A, A_t \in \mathfrak{A}$, $A = \bigcup_{t \in T} A_t$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$, then there exist elements $B_t \in \mathfrak{A}$ $(t \in T)$ such that

$$B_t \subset A_t, \quad B_t \cap B_{t'} = 0 \quad \text{for} \quad t \neq t' \text{ and } A = \bigcup_{t \in T} B_t.$$

Problem 1. Is this theorem true without the hypothesis that $\mathfrak{A}$ is $\mathfrak{m}'$-complete for every $\mathfrak{m}' < \mathfrak{m}$? (**P 434**).

Another problem of this kind is

Problem 2. Find, for every uncountable cardinal $\mathfrak{m}$, a Boolean $\mathfrak{m}$-algebra $\mathfrak{A}$ with the property: if the join $\bigcup_{t \in T} A_t$ exists in $\mathfrak{A}$ and $\overline{\overline{T}} \leqslant \mathfrak{m}$, then there exists a finite subset $T' \subset T$ such that $\bigcup_{t \in T} A_t = \bigcup_{t \in T'} A_t$. (**P 435**).

For $\mathfrak{m} = \aleph_0$ an example of such a Boolean algebra was given by Sierpiński [4].

Problems 3-6 which follow are connected with a classification of Boolean algebras discussed in my paper [7].

Problem 3. Find an example (for every uncountable cardinal $\mathfrak{m}$) of a weakly $\mathfrak{m}$-distributive Boolean $\mathfrak{m}$-algebra which is not $\mathfrak{m}$-distributive (**P 436**).

In the case where $\mathfrak{m} = \aleph_0$ such an example is given by non-atomic measure algebras (i. e. Boolean σ-algebras with a strictly positive finite σ-measure). Other examples can be obtained e. g. by forming the direct