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ON THE COEFFICIENTS OF UNIVALENT FUNCTIONS IN THE UNIT CIRCLE

BY

J. GÓRSKI (CRACOW)

Let

$$w = f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \tag{1}$$

be an analytic univalent function in the unit circle $K: |z| < 1$ and let Δ be the image if K by (1). We denote by D the image of Δ under the transformation $\zeta = 1/w$. D is an unbounded simple connected domain which contains the point $\zeta = \infty$. The complementary set of D to the whole plane is a bounded continuum E whose capacity $d(E)$ is equal to 1.

Let $\eta^{(n)} = \{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-th extremal system of points on E, i. e. a system of n points of E such that

$$\prod_{1 \leqslant j < k \leqslant n} |\eta_j - \eta_k| \geqslant \prod_{1 \leqslant j < k \leqslant n} |\zeta_j - \zeta_k|$$

for every system of n points $\zeta_1, \zeta_2, \ldots, \zeta_n$ of E. It has been proved in [1] that

1° the limits

$$\lim_{n \to \infty} \frac{\eta_1^k + \eta_2^k + \ldots + \eta_n^k}{n} = s_k, \quad k = 1, 2, \ldots,$$

exist,

2° the coefficients b_k of the inverse function $f^{-1}(w)$ are given by the formulas

$$b_{k+1} = \frac{1}{k}(s_k + b_2 s_{k-1} + \ldots + b_k s_1), \quad k = 1, 2, \ldots,$$

3° the coefficients a_2, a_3, a_4 and a_5 of function (1) are given by

$$a_2 = -s_1, \quad a_3 = \frac{3s_1^2 - s_2}{2}, \quad a_4 = \frac{-8s_1^3 + 6s_1 s_2 - s_3}{3},$$

$$a_5 = \frac{-6s_4 + 40s_1 s_3 - 150 s_1^2 s_2 + 125 s_1^4 + 15 s_2^2}{24},$$

Further coefficients a_k can be easily calculated from the identity

$$z \equiv (z+a_2z^2+\ldots)+b_2(z+a_2z^2+\ldots)^2+b_3(z+a_2z^2+\ldots)^3+\ldots$$

Löwner [2] has proved that

$$|b_{k+1}| \leqslant \frac{1\cdot 3\cdot 5\ldots(2k+1)}{1\cdot 2\cdot 3\ldots(k+2)}\cdot 2^{k+1}, \qquad k=1,2,\ldots,$$

the equality holds for Kœbe's function.

We want to prove a number of sharp inequalities and estimations for some expressions containing the coefficients and limits s_k.

1. Observe that $|b_2| = \max$ implies $|a_2| = \max$ and conversely from $|a_2| = \max$ follows $|b_2| = \max$. The maximum is achieved only for Kœbe's function.

Indeed, $b_2 = s_1$ and $a_2 = -s_1$. Therefore $\max|b_2| = \max|a_2|$ and it is known that if $|a_2| = 2$, then $|a_n| = n$, $n = 2, 3, \ldots$

2. $|s_2| \leqslant 6$, the equality holds only for Kœbe's function.

Indeed, according to the area principle we have $|a_2-a_3| \leqslant 1$, and the equality holds only for Kœbe's function. On the other hand,

$$s_2 = s_1^2+s_2(S),$$

where $s_2(S)$ is the value of the second limit s_2 when the coordinate origin is situated at the center of gravity S of the natural mass-distributional on E. Therefore

$$|a_2^2-a_3| = \left|s_1^2-\frac{3s_1^2-s_1^2-s_2(S)}{2}\right| = \frac{|s_2(S)|}{2} \leqslant 1.$$

From the last sharp inequality it follows that $|s_2(S)| \leqslant 2$ and $|s_2| \leqslant |s_1^2|+|s_2(S)| \leqslant 6$. Observe that

$$|b_3| \leqslant |s_1^2|+\frac{|s_2(S)|}{2} \leqslant 5,$$

with equality only for Kœbe's function. If $|b_3| = 5$, then $s_1 = 2e^{i\alpha}$, $s_2(S) = 2e^{2i\alpha}$, $(s_2 = 6e^{2i\alpha})$.

3. By elementary calculation we obtain

$$(*) \qquad s_n = s_n(S)+ns_{n-1}(S)\cdot s_1+\binom{n}{2}s_{n-2}(S)\cdot s_1^2+\ldots+s_1^n,$$

where $s_n(S)$ is defined analogically as $s_2(S)$.

We shall prove that

$$|s_3-s_3(S)| \leqslant 20,$$

with equality only for Kœbe's function.

Indeed, from $(*)$ it follows that

$$|s_3-s_3(S)| = |3s_2(S)\cdot s_1+s_1^3| \leqslant 3\cdot 2\cdot 2+2^3 = 20,$$

with equality holding only for $|s_1| = 2$, i. e. for Kœbe's function. Observe that for Kœbe's function we have $s_3(S) = 0$ and $s_3 = 20$.

4. The limit $s_3(S)$ satisfies the sharp inequality $|s_3(S)| \leqslant 2$. This result follows from the inequality obtained by Garabedian and Schiffer [3]

$$\frac{|s_3(S)|}{3} = |-a_4+2a_2a_3-a_2^3| \leqslant \tfrac{2}{3}.$$

5. $|a_3+b_3| \leqslant 8$ and $|a_3-b_3| \leqslant 2$, with equalities only for Kœbe's function.

Indeed, $a_3+b_3 = 2s_1^2$ and $a_3-b_3 = -s_2(S)$.

6. $|b_4-s_3(S)/3| \leqslant 14$, with equality only for Kœbe's function.

Indeed, from 2° and $(*)$ follows

$$\left|b_4-\frac{s_3(S)}{3}\right| = |s_1|\cdot\left|\frac{3s_2(S)}{2}+s_1^2\right| \leqslant 6+8 = 14.$$

7. $|a_4+b_4| \leqslant 10$ and $|a_4-b_4+\frac{2}{3}s_3(S)| \leqslant 18$, with equality only for Kœbe's function.

Indeed, using formulas 2° and $(*)$ we get

$$|a_4+b_4| = \tfrac{5}{2}|s_1s_2-s_1^3| \leqslant 5|s_2(S)| \leqslant 10,$$

$$|a_4-b_4+\tfrac{2}{3}s_3(S)| = |s_1|\cdot\left|-2s_1^2-\frac{s_2(S)}{2}\right| \leqslant 2(8+1) = 18.$$

8. $|a_5+b_5-2s_1s_3(S)| \leqslant 47$, with equality only for Kœbe's function.

Indeed,

$$a_5 = -\tfrac{1}{4}s_4(S)+s_1^4+\tfrac{2}{3}s_1s_3(S)-\tfrac{3}{2}s_2(S)s_1^2+\tfrac{5}{8}s_2^2(S),$$

$$b_5 = \tfrac{1}{4}s_4(S)+s_1^4+\tfrac{4}{3}s_1s_3(S)+3s_2(S)s_1^2+\tfrac{1}{8}s_2^2(S)$$

and

$$|a_5+b_5-2s_1s_3(S)| \leqslant 2|s_1^4|+\tfrac{3}{2}|s_1^2||s_2(S)|+\tfrac{3}{4}|s_2^2(S)| \leqslant 47.$$

9. If we compare Löwners formulas for the coefficients a_n with s_n we get the following inequalities:

$$s_1 = 2\int_0^\infty e^{-\tau}k(\tau)\,d\tau, \qquad |s_1| \leqslant 2,$$

$$s_2 = 4\Big(\int_0^\infty e^{-\tau}k(\tau)\,d\tau\Big)^2 + 4\int_0^\infty e^{-2\tau}k^2(\tau)\,d\tau; \qquad |s_2| \leqslant 4+\tfrac{4}{2} = 6,$$

$$s_2(S) = 4\int_0^\infty e^{-2\tau}k^2(\tau)\,d\tau, \qquad |s_2(S)| \leqslant 4\tfrac{1}{2} = 2,$$

$$s_3 = 8\Big(\int_0^\infty e^{-\tau}k(\tau)\,d\tau\Big)^3 + 24\int_0^\infty e^{-\tau}k(\tau)\,d\tau\int_0^\infty e^{-2s}k^2(s)\,ds +$$

$$+6\int_0^\infty e^{-3\tau}k^3(\tau)\,d\tau - 12\int_0^\infty e^{-\tau}k(\tau)\Big(\int_\tau^\infty e^{-2s}k^2(s)\,ds\Big)\,d\tau,$$

$$s_3(S) = 6\int_0^\infty e^{-3\tau}k^3(\tau)\,d\tau - 12\int_0^\infty e^{-\tau}k(\tau)\Big(\int_0^\infty e^{-2s}k^2(s)\,ds\Big)\,d\tau.$$

In particular, one obtains the sharp inequalities:

$$|s_3| = \Big|8\Big(\int_0^\infty e^{-\tau}k(\tau)\,d\tau\Big)^3 + 12\int_0^\infty e^{-\tau}k(\tau)\,d\tau\int_0^\infty e^{-2s}k^2(s)\,ds +$$

$$+6\int_0^\infty e^{-3\tau}k^3(\tau)\,d\tau + 12\int_0^\infty e^{-\tau}k(\tau)\Big(\int_\tau^\infty e^{-2s}k^2(s)\,ds\Big)\,d\tau\Big|$$

$$\leqslant 8+\frac{12}{2}+\frac{6}{3}+12\int_0^\infty e^{-\tau}\Big(\frac{e^{-2\tau}}{-2}+\frac{1}{2}\Big)\,d\tau$$

$$= 16+12\Big[-\frac{1}{6}+\frac{1}{2}\Big] = 20,$$

$$|b_4| \leqslant \frac{2|s_3|+3|s_1|\cdot|s_2|+|s_1^3|}{6} \leqslant \frac{2\cdot 20+3\cdot 2\cdot 6+8}{6} = 14.$$

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