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# ON THE PARTIAL SUMS OF CERTAIN LAURENT EXPANSIONS

BY

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**1. Introduction.** In this note* we are concerned with the partial sums

(1) $$S_n(z) = z + \sum_{1}^{n} \frac{a_k}{z^k},$$

of functions

(2) $$f(z) = z + \sum_{1}^{\infty} \frac{a_k}{z^k}, \quad z = Re^{i\theta},$$

which are analytic in the domain $D$: $|z| > 1$. We obtain some results which are reminiscent of earlier ones obtained by Szegö [16] who considered functions analytic in the unit disc. Whereas all of Szegö's results are sharp, none of ours are sharp. Hence we leave unanswered several interesting and important questions.

**2. Univalent functions.** In an earlier note [13] we offered a slight improvement of a result due to Kung Sun; we proved that there exists a constant $R_1$ such that if $f(z)$ in (2) is univalent for $|z| > 1$, then each partial sum (1) is univalent for $|z| > R_1$, and we showed that $[3/2]^{1/4} \leqslant R_1 \leqslant \sqrt{2}$. Now if we consider the univalent function

$$g(z) = z + \frac{4ie^{-3}}{z} + \frac{\frac{1}{2} + e^{-6}}{z^3} + \dots$$

discovered by Garabedian and Schiffer ([2], p. 133) while investigating a certain coefficient problem, then we find that the derivative of the partial sum

$$z + \frac{4ie^{-3}}{z} + \frac{\frac{1}{2} + e^{-6}}{z^3}$$

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has a zero on the circle $|z| = [\frac{3}{2}+3e^{-6}]^{1/4}$. Hence our constant $R_1$ must satisfy $[\frac{3}{2}+3e^{-6}]^{1/4} \leqslant R_1 \leqslant \sqrt{2}$.

**3. Star-like function.** The upper bound $\sqrt{2}$ on $R_1$ noted above was obtained by straightforward, classic methods. However, a recent elegant result due to Pomerenke ([9], p. 274) and Robertson ([15], p. 516) permits us to show a deeper significance to the number $\sqrt{2}$. First we quote the result due to Pomerenke and Robertson.

THEOREM 1. *If $f(z)$ in (2) is analytic for $|z|>1$, and if*

$$\sum_{1}^{\infty} \frac{k\,|a_k|}{R^{k+1}} \leqslant 1,$$

*then $f(z)$ maps $|z|>r$ univalently onto a domain that is star-like with respect to the origin in the image plane. If*

$$\sum_{1}^{\infty} \frac{k^2\,|a_k|}{R^{k+1}} \leqslant 1,$$

*then $f(z)$ maps $|z|>R_n$ univalently onto a convex domain.*

We now have the following results.

THEOREM 2. *If $f(z)$ in (2) is analytic for $|z|>1$ and maps $|z|>1$ univalently onto a domain that is star-shaped with respect to the origin in the image plane, then there is a constant $R_2$ such that each partial sum (1) maps $|z|>R_2$ onto a domain that is star-shaped with respect to the origin in the image plane; here $[\frac{3}{2}]^{1/4} \leqslant R_2 \leqslant \sqrt{2}$.*

Proof. For the lower bound $[\frac{3}{2}]^{1/4}$, we need but consider $S_3(z)$ for the univalent and star-like function

$$z\left(1+\frac{1}{z^4}\right)^{1/2} = z+\frac{1}{2z^3}+\ldots \tag{3}$$

To obtain the upper bound we need but use the well-known "area principle" and the Schwarz inequality to obtain

$$\left[\sum_{1}^{n} \frac{k\,|a_k|}{R^{k+1}}\right]^2 \leqslant \sum_{1}^{n} k\,|a_k|^2 \sum_{1}^{n} \frac{k}{R^{2k+2}} < \sum_{1}^{\infty} \frac{k}{R^{2k+2}} = \frac{1}{(R^2-1)^2},$$

after which an appeal to Theorem 1 yields the desired result.

We can obtain a rough estimate of the radius $\varrho_n^*$ of star-likeness of each partial sum $S_n(z)$ as a function of $n$. If we apply standard distortion theorems for star-like mappings to $f(z)$, $S_n(z)$ and to $Q_n(z) = f(z) - S_n(z)$, then we obtain



$$\operatorname{Re}\frac{zS_n'(z)}{S_n(z)} \geqslant \operatorname{Re}\frac{zf'(z)}{f(z)} - \frac{\left|\dfrac{zf'(z)}{f(z)}Q_n(z) - zQ_n'(z)\right|}{|f(z)-Q_n(z)|} \tag{4}$$

$$\geqslant \frac{R^2-1}{R^2+1} - \frac{R^2}{R^2-1}\,\frac{(R^2+1)+\sqrt{(n+2)R^2-(n+1)}}{R^{n+1}(R^2-1)^{3/2}-R^2}, \qquad |z| = R.$$

It is known that a necessary and sufficient condition that $w = S_n(z)$ map a circle $|z| = R$ onto a star-like curve (with respect to the origin, presumably inside the curve) is that the left hand member of (4) be positive [3]. Hence we can obtain, after a tedious and not-too-ingenious computation, the result that $w = S_n(z)$ maps

$$|z| > \varrho_n = 1+\frac{1}{2(n+1)}\log(n+2)$$

onto a domain that is star-like with respect to the origin.

**4. Convex maps.** Star-like maps naturally lead to considerations of convex maps. The techniques are too-well known [3]; so we content ourselves with a brief description of our results.

THEOREM 3. *If the function $f(z)$ in (2) is analytic and univalent for $|z|>1$, and if $f(z)$ maps $|z|>1$ onto a convex domain, then there is a constant $R_3$ such that each partial sum (1) maps $|z|>R_3$ onto a convex domain; here $(\frac{3}{2})^{1/4} < R_3 < 2$.*

Proof. Again, the Schwarz inequality and the Bieberbach "area principle" yield

$$\left[\sum_{1}^{n} \frac{k^2\,|a_k|}{R^{k+1}}\right]^2 \leqslant \sum_{1}^{n} \frac{k^3}{R^{2k+2}} < \sum_{1}^{\infty} \frac{k^3}{R^{2k+2}} = \frac{R^4+4R^2+1}{(R^2-1)^4}. \tag{5}$$

Another application of Theorem 1 yields the upper estimate given. Consideration of the convex function associated with the star-like function (3)

$$h(z) = \int \left(1+\frac{1}{z^4}\right)^{1/2} dz = z - \frac{1}{6z^3}+\ldots \tag{6}$$

shows that the third partial sum of (6) fails even to be univalent for $|z| \leqslant [\frac{3}{2}]^{1/4}$, which leads to the lower bound on $R_3$.

It may be worth remarking that in (5) we did not make use of the convexity of the function $f(z)$; hence we have the result that for any function (2), analytic and univalent for $|z|>1$, all partial sums (1) are univalent and convex functions for $|z|>\sqrt{2}$.

**5. Functions whose derivative has a positive real part.** A pretty result due to Wolff, Noshiro and Warschawski [6] states that if a function is analytic in a convex domain, and if its derivative has a positive real part in that domain, then the function is univalent there. This leads quite naturally to a consideration of functions (2) for which $\operatorname{Re} f'(z)$ is positive. If $f(z)$ in (2) is analytic for $|z|>1$, then we are concerned with

$$f'(z) = 1 - \frac{a_1}{z^2} - \frac{2a_2}{z^3} - \dots \tag{7}$$

for which $\operatorname{Re} f'(z) > 0$ holds for $|z|>1$.

First we note that the function

$$\Phi(z) = z - \log\frac{1+1/z}{1-1/z} = z - \frac{2}{z} - \frac{2}{3z^3} - \dots \tag{8}$$

satisfies the conditions stated in the preceding paragraph. Simple calculations show that $\Phi(z)$ vanishes for $z=\pm a$, where $a$ is the positive root of the equation

$$a = \log\frac{a+1}{a-1},$$

and that $1{,}54 < a < 1{,}55$.

However, we can prove the following weak result:

THEOREM 4. *If $f(z)$ in* (2) *is analytic for $|z|>1$, and if its derivative* (7) *has a positive real part for $|z|>1$, then there is a constant $R_4$, such that $f(z)$ and each of its partial sums* (1) *are univalent for $|z|>R_4$; here* $1{,}54 < R_4 < 2$.

Proof. Since (7) has positive real part for $|z|>1$, the often used Carathéodory-Töplitz inequalities subsist [3]:

$$|ka_k| \leqslant 2, \quad k = 1, 2, \dots \tag{9}$$

If we choose $z_1, z_2$, with $|z_1| = |z_2| = R > 1$, then the most elementary calculations and (9) yield

$$\left|\frac{f(z_2)-f(z_1)}{z_2-z_1}\right| \geqslant 1 - \sum_1^\infty \frac{k|a_k|}{R^{k+1}} \geqslant 1 - \frac{2}{R(R-1)}. \tag{10}$$

From (10) we conclude that $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$, $|z_1| = |z_2|$, provided $|z_1| = |z_2| > 2$. Hence each circle $|z| = R > 2$ is mapped onto a simple closed curve by $f(z)$ and so $f(z)$ is univalent for $|z|>2$.

The same sort of computation holds for the partial sums $S_n(z)$. This completes the consideration of the upper bound for $R_4$. To complete the proof of the theorem we need but refer to the function $\Phi(z)$ in (8) to obtain the lower bound on $R_4$.



We must mention that application of Theorem 1 to functions of the type under consideration yields the results (a) $f(z)$ and all partial sums are star-like for $|z|>2$, and (b) $f(z)$ and all partial sums are convex functions for $|z| > \sqrt{2}+1$.

**6. Concluding remarks.** We have carried out studies of the type contained in this note for other classes of *functions $f(z)$ of the form* (2); for example, we have studied (i) functions with bounded boundary rotation studied by Paatero [7, 8], (ii) functions convex in one direction, introduced by Robertson [14, 17], (iii) close-to-convex functions, studied by Kaplan [4], Pomerenke [9], Robertson [15], Umezawa [18], and the present author [10], and (iv) close-to-star functions studied by the present author [11]. Our results are of the same general character as the results contained in this note; they will be exhibited elsewhere.

If we return to the functions of section 5, those having a derivative with positive real part for $|z|>1$, then we can obtain at least one Herglotz representation [3] that leads to the following sequence of obvious steps:

$$f'(z) = \frac{1}{2\pi}\int_0^{2\pi} \left[\left(1+\frac{e^{-i\varphi}}{z}\right)\Big/\left(1-\frac{e^{-i\varphi}}{z}\right)\right] d(\mu\varphi)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[1+2\sum_1^\infty\left(\frac{e^{-i\varphi}}{z}\right)^k\right] d\mu(\varphi),$$

$$f(z_2)-f(z_1) = (z_2-z_1) + \frac{1}{2\pi}\int_0^{2\pi} 2e^{-i\varphi}\log\frac{1-e^{-i\varphi}/z_2}{1-e^{-i\varphi}/z_1}\, d\mu(\varphi), \tag{11}$$

$$f(z) = z + \frac{1}{2\pi}\int_0^{2\pi} 2e^{-i\varphi}\log\left(1-\frac{e^{-i\varphi}}{z}\right) d\mu(\varphi).$$

To finally achieve (11) we have used the fact that $f'(z)$ has the form (7) so that the monotone increasing function $\mu(\varphi)$ satisfies the two relations

$$\int_0^{2\pi} d\mu(\varphi) = 2\pi, \qquad \int_0^{2\pi} e^{-i\varphi} d\mu(\varphi) = 0.$$

Now the most elementary calculations show that $f'(z)$ is subordinate to $[(z^2+1)/(z^2-1)]$ for $|z|>1$, so that the inequalities

$$\frac{|z|^2-1}{|z|^2+1} \leqslant |f'(z)| \leqslant \frac{|z|^2+1}{|z|^2-1}, \quad |z|>1,$$

$$\operatorname{Re} f'(z) \geqslant \frac{|z|^2-1}{|z|^2+1}, \quad |z|>1,$$

must hold [5], p. 167). Moreover, from (11) we obtain

$$|z|+2\log\left(1-\frac{1}{|z|}\right) \leqslant |f(z)| \leqslant |z|-2\log\left(1-\frac{1}{|z|}\right),$$

which is *not* a sharp result. Finally it seems that the function $\Phi(z)$ in (8) should be the "extremal" for the considerations of section 5; this we have not been able to establish.

It appears that one of the obstacles to obtaining sharper results in the considerations contained in this note is the absence of a unique over-riding "extremal" function, analogous to the Koebe function $[z/(1-z)^2]$ for similar considerations of functions analytic in the unit disc; each partial $S_n(z)$ seems to require an extremal function peculiar to that partial sum, much as Clunie found in his study of the coefficient problem for star-like functions [1].

Finally, the results contained in this note were communicated to the International Congress of Mathematicians, Stockholm, 1962 [12]. We had hoped to obtain sharper results for the Cracow Conference, but we were unsuccessful. It would certainly be more than desirable to obtain the "best" possible answers to the questions raised here.

Added in proof. Professor Jan Krzyż has called our attention to a recent note by V. G. Lozovik, in Dopovidi Akad. Nauk URSR, 1962, pp. 856-858, in which there appear some results that complement and overlap some of those contained in sections 5 and 6 above.

## *REFERENCES*

[1] J. Clunie, *On meromorphic schlicht functions*, Journal of the London Mathematical Society 34 (1959), p. 215-216.

[2] P. Garabedian and M. Schiffer, *A coefficient inequality for schlicht functions*, Annals if Mathematics 61 (1955), p. 116-136.

[3] G. M. Golusin, *Geometrische Funktionentheorie*, Berlin 1957.

[4] W. Kaplan, *Close-to-convex functions*, Michigan Mathematical Journal 1 (1952), p. 169-185.

[5] J. E. Littlewood, *Theory of functions*, Oxford 1944.

[6] T. H. MacGregor, *Functions whose derivative has a positive real part*, Transactions of the American Mathematical Society 104 (1962), p. 532-537.

[7] V. Paatero, *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, Annales Academiae Scientiarum Fennicae, A, 1931.

[8] — *Über Gebiete von beschränkter Randdrehung*, ibidem A, 1933.

[9] C. Pomerenke, *Über einige Klassen meromorpher schlichter Funktionen*, Mathematische Zeitschrift 78 (1962), p. 263-284.

[10] M. O. Reade, *Some remarks on schlicht functions*, Proceedings of the International Mathematical Congress, Amsterdam 1954, vol. 1, p. 462.

[11] — *On close-to-convex functions*, Michigan Mathematical Journal 3 (1955-1956) p. 59-62.



[12] — *On sections of certain Laurent series*, Abstracts, International Mathematical Congress, Stockholm 1962, p. 107.

[13] — *On the partial sums of certain Laurent expansions*, Journal of the Mathematical Society of Japan 15 (1963), p. 66-68.

[14] M. S. Robertson, *Analytic functions star-like in one direction*, American Journal of Matematics 58 (1936), p. 465-472.

[15] — *Convolutions of schlicht functions*, Notices — American Mathematical Society 8 (1961), p. 516.

[16] G. Szegö, *Zur Theorie der schlichten Abbildungen*, Mathematische Annalen 100 (1928), p. 188-211.

[17] T. Umezawa, *Analytic functions convex in one direction*, Journal of the Mathematical Society of Japan 4 (1952), p. 194-202.

[18] — *On the theory of univalent functions*, Tohôku Mathematical Journal 7 (1955), p. 212-223.

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