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ON SOME QUESTIONS CONCERNING LACUNARY POWER SERIES OF TWO VARIABLES

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Some years ago I proved (1) the following

THEOREM 1. *Let $f(z)$ be a transcendental entire function. Let $Z_a(n)$ and $Z_b(n)$ denote the number of vanishing coefficients among the first $n+1$ coefficients of the power series expansion of $f(z)$ around the points a and b $(a \neq b)$, respectively $(n = 0, 1, \ldots)$. Then we have*

$$\varlimsup_{n\to\infty} \frac{Z_a(n)+Z_b(n)}{n} \leqslant 1.$$

This theorem contains as a special case the

THEOREM 2. *The power series expansions of a transcendental entire function around two different points can not have both Fabry gaps.*

The purpose of the present paper* is to consider some consequences of these results for power series of two variables.

The first question is now: how to define the lacunarity for power series of two variables in analogy with the lacunarity in Fabry's sense for power series of one variable. Below we give different reasonable definitions:

Let

$$f(z, w) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l}(z-z_0)^k(w-w_0)^l, \tag{1}$$

$$\varepsilon_{k,l} = \begin{cases} 0 & \text{if} \quad a_{k,l} = 0 \\ 1 & \text{otherwise.} \end{cases} \qquad (k, l = 0, 1, \ldots) \tag{2}$$

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* Presented to the Third Conference on Analytic functions held in Cracow, 30. VIII - 4. IX. 1962.

(1) C. Rényi, *On a conjecture of G. Pólya*, Acta Mathematica Academiae Scientiarum Hungaricae 7 (1956), p. 145-150.

We say, that the series (1) is *lacunary in the sense* A, if

$$\lim_{\substack{N\to\infty\\ M\to\infty}} \frac{\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\varepsilon_{k,l}}{N.M} = 0.$$

We say, that the series (1) is *lacunary in the sense* A^*, if

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\varepsilon_{k,l}}{N^2} = 0$$

The lacunarity A implies trivially A^*, but not conversely: for instance if

$$\varepsilon_{k,l} = \begin{cases} 0 & \text{if} \quad l \leqslant k^2 \\ 1 & \text{otherwise,} \end{cases} \qquad (k, l = 0, 1, \ldots)$$

then

$$\frac{\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\varepsilon_{k,l}}{N^2} = O\left(\frac{1}{\sqrt{N}}\right),$$

but

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N-1}\sum_{l=0}^{N^3-1}\varepsilon_{k,l}}{N^4} = 1.$$

We say, that the series

(3) $$f(z, w) = \sum_{k=0}^{\infty} b_k(w)(z-z_0)^{n_k}$$

resp.

(4) $$f(z, w) = \sum_{k=0}^{\infty} c_k(z)(w-w_0)^{n_k},$$

is *lacunary in the sense* B *resp.* C if

$$\lim_{k\to\infty} \frac{k}{n_k} = 0.$$

It is obvious, that the lacunarity B resp. C implies A, but not conversely.

We say, that the series (1) is *lacunary in the sense* D if

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N-1}\delta_n}{N} = 0,$$

where

$$\delta_n = \begin{cases} 0 & \text{if} \quad \sum_{k=0}^{n}\varepsilon_{k,n-k} = 0, \\ 1 & \text{otherwise,} \end{cases} \qquad (n = 0, 1, \ldots)$$

and $\varepsilon_{k,l}$ are defined by (2). It is evident, that the lacunarity D implies A, but not conversely and also, that neither D implies B resp. C, nor conversely.

Now let us see what we can say about consequences of the theorems 1 and 2 for power series of two variables, which are lacunary in one of the above defined senses.

Lacunarity D. The following question presents itself: does there exist an entire function $f(z, w)$, which is not a polynomial in any of its variables, such that its power series expansions around two different points (z_0, w_0) and (z_1, w_1) should be both lacunary in the sense D? The answer is: yes. For example

(5) $$f(z, w) = \sum_{k=0}^{\infty} \frac{(z-w)^{k^2}}{k^2!} = \sum_{k=0}^{\infty} \frac{[(z-a)-(w-a)]^{k^2}}{k^2!}$$

is such a function (its power series around any point (a, a) is lacunary in the sense D).

Consequently we have to introduce some complementary condition, namely the

CONDITION 1. For any constants α and β there exists at least one n such that $[\partial^n f/\partial w^n]_{w=\alpha z+\beta}$ is a transcendental entire function of z.

Remark 1. Condition 1 can be replaced by the following: for any constants α and β there exists at least one n such that $[\partial^n f/\partial z^n]_{z=\alpha w+\beta}$ is a transcendental entire function of w.

Remark 2. If $f(z, w)$ does not satisfy condition 1 for some constants α, β, then it is of the form

$$f(z, w) = \sum_{n=0}^{\infty} (w-\alpha z-\beta)^n P_n(z),$$

where $P_n(z)$ is a polynomial of z $(n = 0, 1, \ldots)$.

And now we have

THEOREM I. *Let $f(z, w)$ be a function satisfying condition* 1, *(z_0, w_0) and (z_1, w_1) two points for which $z_0 \neq z_1$ and $w_0 \neq w_1$. Then the power series expansions of $f(z, w)$ around the points (z_0, w_0) and (z_1, w_1) cannot be both lacunary in the sense* D.

Proof. 1. We may suppose $z_0 = w_0 = 0$, $z_1 = w_1 = 1$. As a matter of fact if the assertion of Theorem I would not be valid for the function

$f(z,w)$ in the points (z_0, w_0) and (z_1, w_1) $(z_0 \neq z_1,\ w_0 \neq w_1)$, then it would not be valid for the function $g(z,w) = f[z_0+(z_1-z_0)z,\ w_0+(w_1-w_0)w]$ at the points $(0,0)$ and $(1,1)$; further if $f(z,w)$ satisfies condition 1, then $g(z,w)$ does also.

2. We introduce the notations $\partial^n f(z,w)/\partial w^n = h_n(z,w)$ and $h_n(z,z) = \varphi_n(z)$ $(n = 0, 1, \ldots)$.

Let us suppose, that the assertion of Theorem I is not true; this means that the power series expansions of $f(z,w)$ around the points $(0,0)$ and $(1,1)$ are both lacunary in the sense D. In this case the power series expansions around the points $(0,0)$ and $(1,1)$ of any of the functions $h_n(z,w)$ would be lacunary in the sense D and the Taylor series around the points 0 and 1 of the corresponding functions $\varphi_n(z)$ would possess Fabry gaps. But by the condition 1 there exists at least one $n = n_0$ such, that $\varphi_{n_0}(z)$ is a transcendental entire function, which contradicts Theorem 2.

Remark 3. The condition $z_0 \neq z_1$ and $w_0 \neq w_1$ is essential: the function

$$f(z,w) = \sum_{k=0}^{\infty} \frac{z^{k^3} w^k}{k^3!} \tag{6}$$

satisfies condition 1 and its power series expansions around any point $(0, a)$ is lacunary in the sense D.

Remark 4. Employing Theorem 1 instead of Theorem 2, we may get sharper results.

Lacunarity B and C. Since the cases B and C are symmetric, we shall consider only the lacunarity B.

We shall consider only entire functions $f(z,w)$ satisfying the

CONDITION 2. $\partial^{n+m} f(z,w)/\partial z^n \partial w^m \not\equiv 0$ for $n, m = 0, 1, \ldots$

Remark 5. If $f(z,w)$ does not satisfy condition 2, then it is of the form

$$f(z,w) = \sum_{k=0}^{\infty} z^k \Big(\sum_{l=0}^{l_0} a_{k,l} w^l\Big) + \sum_{l=l_0+1}^{\infty} w^l \Big(\sum_{k=0}^{k_0} a_{k,l} z^k\Big).$$

Let

$$f(z,w) = \begin{cases} \sum_{k=0}^{\infty} b_k(w)(z-z_0)^{n_k}, \\ \sum_{n=0}^{\infty} b_n^*(w)(z-z_1)^n \end{cases} \tag{7}$$

and let us put

$$\Delta_n = \begin{cases} s & \text{if} \quad b_n^*(w) \text{ is a polynomial of degree } s, \\ \infty & \text{if} \quad b_n^*(w) \text{ is transcendental.} \end{cases} \quad (n, s = 0, 1, \ldots) \tag{8}$$

If $f(z,w)$ is an entire function satisfying condition 2, then obviously there exists some set H of natural numbers such, that $\lim\limits_{\substack{n \in H \\ n\to\infty}} \Delta_n = \infty$. But much more is true. We have the following

THEOREM II. *Let $f(z,w)$ be an entire function satisfying condition 2, the power series expansions (7) of which around the point z_0 is lacunary in the sense* B *and let $z_1 \neq z_0$. Then — using the notations (7) and (8) — there exists a set H of natural numbers the upper density of which is 1, such that* $\lim\limits_{\substack{n \in H \\ n\to\infty}} \Delta_n = \infty$.

Proof. Let w_0 be a fixed value, for which

$$\left[\frac{d^j b_k(w)}{dw^j}\right]_{w=w_0} \neq 0, \tag{9}$$

except for those values of j and k for which

$$\frac{d^j b_k(w)}{dw^j} \equiv 0 \quad (k = 0, 1, \ldots;\ j = 0, 1, \ldots),$$

and

$$\left[\frac{d^j b_n^*(w)}{dw^j}\right]_{w=w_0} \neq 0, \tag{10}$$

except for those values of j and n for which

$$\frac{d^j b_n^*(w)}{dw^j} \equiv 0 \quad (n = 0, 1, \ldots;\ j = 0, 1, \ldots).$$

Let us denote by $Z^{(j)}(n)$ the number of zeros in the series $b_0^{*(j)}(w_0)$, $b_1^{*(j)}(w_0), \ldots, b_{n-1}^{*(j)}(w_0)$ $(j = 0; 1, \ldots;\ n = 1, 2, \ldots)$. In other words,

$$Z^{(j)}(n) = \sum_{\substack{k<n \\ \Delta_k < j}} 1.$$

Let us denote by $N^{(j)}$ the set of those natural numbers n, for which

$$Z^{(j)}(n) < \frac{n}{2^{j+1}} \qquad (j = 0, 1, \ldots).$$

It follows from (9) and Theorem 1, that each of the sets $N^{(j)}$ is infinite. Further (10) ensures that $N^{(j)} \subset N^{(j-1)}$.

We choose from the set of natural numbers a subset $\{\nu_j\}$ in the following way: let ν_0 be the least element of $N^{(0)}$ and ν_j the least element of $N^{(j)}$ satisfying $\nu_j > \nu_{j-1}$ $(j = 1, 2, \ldots)$. Now we define the set H of natural numbers in the following manner: $n \in H$ if $\nu_{j-1} < n \leqslant \nu_j$ and $\Delta_n \geqslant j$ $(j = 1, 2, \ldots)$. Then obviously

$$\lim_{\substack{n \in H \\ n \to \infty}} \Delta_n = \infty.$$

On the other hand, the upper density of H is 1. As a matter of fact, if we introduce the notation

$$H(N) = \sum_{\substack{n \in H \\ n \leqslant N}} 1;$$

we have

$$H(\nu_j) \geqslant \nu_j - Z^{(j)}(\nu_j),$$

$$\frac{H(\nu_j)}{\nu_j} \geqslant 1 - \frac{Z^{(j)}(\nu_j)}{\nu_j} > 1 - \frac{1}{2^{j+1}}, \quad (j = 1, 2, \ldots)$$

and thus

$$\overline{\lim_{N \to \infty}} \frac{H(N)}{N} = 1.$$

Remark 6. Theorem II implies that the power series expansions (7) with $z_0 \neq z_1$ cannot both possess B gaps.

Remark 7. The assertion of Theorem II is also valid if not the power series expansion of the function itself, but of one of its derivatives with respect to w has B gaps in the point z_0.

Remark 8. The coefficients $b_k^*(w)$ and $b_n(w)$ can be all polynomials. For example let us consider the function

$$f(z, w) = \sum_{l=0}^{\infty} c_l z^{l^3} (z-1)^l P_l(w)$$

where $P_l(w)$ are polynomials, the degrees of which are not bounded. In the point $z = 0$ we have B gaps, namely

$$\text{coeff. } z^s = \begin{cases} c_l (-1)^{l-j} \binom{l}{j} P_l(w) & \text{if} \quad s = l^3 + j,\ j = 0, 1, \ldots, l, \\ 0 & \text{otherwise.} \end{cases}$$

At the same time at the point $z = 1$ we have

$$\text{coeff.} (z-1)^s = \sum_{l \leqslant s \leqslant l^3 + l} c_l \binom{l^3}{s-l} P_l(w), \quad s = 0, 1, \ldots$$

Lacunarity A. As regards lacunarity A we have at present only negative results. It is easy to see that the power series expansions of the function

(11) $$f(z, w) = \sum_{k=0}^{\infty} \frac{(zw)^{k^2}}{k^2!}$$

around any points $(a, 0)$ and $(0, b)$ are lacunary in the sense A.

Perhaps it is true that if there exist five points (z_j, w_j), $(j = 1, \ldots, 5)$ such that the power series expansions of a non-polynomial entire function around each of these points possess A gaps, then three of them satisfy some equation

$$\alpha z + \beta w + \gamma = 0$$

with $|\alpha| + |\beta| + |\gamma| > 0$.

But this hypothesis is certainly not valid for the

Lacunarity A*. Let r points a_i resp. b_j $(i, j = 1, 2, \ldots, r)$ be given arbitrarily in the z resp. w plane. The power series expansions of the function

(12) $$f(z, w) = \sum_{m=0}^{\infty} c_m \prod_{i=1}^{r} (z - a_i)^{(r+2)^m} \prod_{j=1}^{r} (z - b_j)^{(r+1)^m}$$

around any point (a_i, b_j) are lacunary in the sense A*. As a matter of fact, in the expansion of $f(z, w)$ around some point (a_i, b_j) there are present only such terms $(z - a_i)^s (z - b_j)^t$, for which

$$\begin{aligned} (r+2)^m &\leqslant s \leqslant r(r+2)^m, \\ (r+1)^m &\leqslant t \leqslant r(r+1)^m, \end{aligned} \quad (m = 0, 1, \ldots)$$

and therefore if $r(r+2)^n < N \leqslant r(r+2)^{n+1}$ $(n = 1, 2, \ldots)$, then

$$\frac{\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \varepsilon_{k,l}}{N^2} < \frac{\sum_{m=0}^{n+1} r^2 (r+2)^m (r+1)^m}{r^2 (r+2)^{2n}} = O\left[\left(\frac{r+1}{r+2}\right)^n\right].$$

Let now be given arbitrarily denumerable sets of points a_i and b_j $(i, j = 1, 2, \ldots)$. In a similar way as before it is easy to see that the power series expansions of the function

(13) $$f(z, w) = \sum_{m=0}^{\infty} c_m \prod_{i=1}^{m} (z - a_i)^{(m+2)^m} \prod_{j=1}^{m} (z - b_j)^{(m+1)^m}$$

around any point (a_i, b_j) are lacunary in the sense A*.

Remark 9. In the case of examples (5), (6), (11), (12) and (13) it is not essential that they are entire functions: big coefficients have no influence on the fact that the power series expansions of the corresponding functions will be lacunary in this or that sense around many points. While this does not exclude completely that some gap theorem of Fabry's kind is valid for functions of two variables, it shows, that this will certainly not be a trivial generalization of the original theorem for functions of one variable.

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