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SOME APPLICATIONS OF THE METHOD OF EXTREMAL POINTS IN THE THEORY OF ANALYTIC FUNCTIONS OF ONE COMPLEX VARIABLE

BY

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We present here an outline* of some recent results obtained with the aid of Leja's method.

Let R be a topological space, E a bounded closed set and $\omega(x,y)$ a real function of two points $x, y \in R$ which satisfies the following conditions:

$$\omega(x,y) \geqslant 0, \quad \omega(x,y) = \omega(y,x).$$

Denote by $f(x)$ a real function defined on E and consider the product

$$V(p^{(n)}, \omega, \lambda f) \doteq \prod_{1\leqslant i<k\leqslant n} \omega(p_i, p_k) \exp \lambda[f(p_i)+f(p_k)] \tag{1}$$

where $\lambda > 0$ is a parameter and $p^{(n)} = (p_1, p_2, \ldots, p_n)$ an arbitrary system of n points of E.

Let $V_n(\omega, \lambda f)$ be the upper bound of the product (1) when the system $p^{(n)}$ varies in E. When $f(x)$ and $\omega(x,y)$ are continuous functions there exists at least one system of n points $q^{(n)} \in E$ such that

$$V_n(\omega, \lambda f) = V(q^{(n)}, \omega, \lambda f).$$

A system $q^{(n)} = (q_1, \ldots, q_n)$ is called the *n-th extremal system* of points of E with respect to $\omega(x,y)$ and $\lambda f(x)$.

It has been proved by Leja [6] that the limit

$$\lim_{n\to\infty} V_n(\omega, \lambda f)^{2/n(n-1)} = v(\omega, \lambda f)$$

exists. The number $v(\omega, \lambda f) \geqslant 0$ is called the *ecart* of the set E.

Let $\Phi^{(j)}(x, p^{(n)}, \omega, \lambda f)$, $j = 1, \ldots, n$, be the sequence of functions

$$\Phi^{(j)}(x, p^{(n)}, \omega, \lambda f) = \Bigg[\prod_{\substack{k=1\\ k\neq j}}^{n} \frac{\omega(x, p_k)}{\omega(p_j, p_k)}\Bigg] \exp n\lambda f(p_j).$$

* Presented to the Third Conference on Analytic Functions, held in Cracow, 30. VIII-4. IX. 1962.

Put

$$\Phi_n(x, \omega, \lambda f) = \inf_{p^{(n)} \in E} \max_j \Phi^{(j)}(x, p^{(n)}, \omega, \lambda f).$$

The following theorem has been proved in [1]: *If* $v(\omega, \lambda f) > 0$, *then the limit*

$$\Phi(x, \omega, \lambda f) = \lim_{n \to \infty} [\Phi_n(x, \omega, \lambda f)]^{1/n} \tag{2}$$

exists for $x \in R$. The function $\Phi(x, \omega, \lambda f)$ defined by (2) is called the *extremal function* of E with respect to $\omega(x, y)$ and $\lambda f(x)$.

On the other hand, consider the functions $\Phi^{(j)}(s, q^{(n)}, \omega, \lambda f)$ where $q^{(n)}$ is the n-th extremal system of points. Denote by $\Delta_n^{(j)}$, $j = 1, 2, \ldots, n$, the product

$$\Delta_n^{(j)} = \prod_{\substack{k \neq j \\ k=1}}^{n} \omega(p_j, p_k) \exp(-\lambda f(p_k))$$

and suppose that the indices are chosen in such a way that $\Delta_n^{(1)} \leqslant \Delta_n^{(j)}$, $j = 1, 2, \ldots, n$. In [7] the following has been proved:

If $\lim_{n\to\infty} \sqrt[n]{\Delta_n^{(1)}} = v > 0$, *then the limit*

$$\lim_{n \to \infty} \Phi^{(1)}(x, q^{(n)}, \omega, \lambda f)^{1/n} = \Phi(x, \omega, \lambda f)$$

exists for $x \in R - E$.

We present now some applications of the extremal function $\Phi(x, \lambda f)$.

1. *Green's function.* Let λ be 0 and let $\omega(x, y)$ be the distance of two points of the complex plane. Then $\log \Phi(x, \omega, 0)$ is the generalized Green's function for the unbounded component D_∞ of the complement of E with the pole at ∞.

2. *Conformal mapping.* Le θ be a real number such that the function

$$\psi_n(x) = e^{i\theta_n} \Phi^{(1)}(x, q^{(n)}, \omega, 0)^{1/n}$$

is positive at a fixed point $x_0 \in D_\infty$ of the domain D_∞ which is supposed to be simply connected. Then

$$w = \lim_{n \to \infty} \psi_n(x) = \psi(z)$$

is the conformal mapping function of D_∞ onto the circle $|w| > 1$, $\psi(\infty) = \infty$.

3. *Dirichlet's problem.* Let a_i, $i = 1, \ldots, k$, be non-negative and such that $a = \sum_i a_i > 0$. If

$$f(x) = \frac{1}{a} \sum_i a_i f_i(x),$$

then

$$\prod_{i=1}^{k} \Phi^{a_i}(x, \omega, f_i) \leqslant \Phi^{a}(x, \omega, f).$$

This inequality implies the following: *If* $f(x)$ *and* $\tilde{f}(x)$ *are real functions defined on* E *and* $0 < \lambda' < \lambda$, *then*

$$\left[\frac{\Phi(x, \omega, f + \lambda \tilde{f})}{\Phi(x, \omega, f)}\right]^{1/\lambda} \leqslant \left[\frac{\Phi(x, \omega, f + \lambda' \tilde{f})}{\Phi(x, \omega, f)}\right]^{1/\lambda'}.$$

Moreover, the function

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \log \frac{\Phi(x, \omega, f + \lambda \tilde{f})}{\Phi(x, \omega, f)}$$

is harmonic outside E.

In particular: *If* D *is a simply connected bounded domain whose boundary is* E, *then*

$$\lim_{\lambda \to 0} \lambda^{-1} \log \Phi(x, \omega, \lambda f) = u(x)$$

is the solution of the Dirichlet problem for D *with boundary values* $f(x)$.

This result has been generalized to the case of multiply connected domain and to the case of a domain in n-dimensional space with $n \geqslant 3$ (see [9] and [2]).

4. In the case of the 3-dimensional space and of the equation $\Delta u - c^2 v = 0$ the first boundary value problem for a given domain has been solved. It was necessary to choose for $\omega(x, y)$ the function

$$\exp\left\{\lambda [f(x) + f(y)] - \frac{e^{-c|x-y|}}{|x-y|}\right\}.$$

For sufficient smooth boundary of the domain and for the boundary value $f(x)$ which satisfies the Lipschitz condition the second passage to the limit $\lambda \to 0$ is unnecessary [4].

5. *Bremermann's problem.* Let $\omega(x, y)$ be the function

$$|h(x, y)| \exp\{-\lambda [f(x) + f(y)]\},$$

where $h(x, y)$ is an analytic function of two points $x = (z_1, z_2)$, $y = (\xi_1, \xi_2)$ defined in a domain D with the boundary E in the space of two complex variables. Using the method of extremal points Górski [5] constructed a plurisubharmonic function

$$u_\lambda(x) = \lambda^{-1} \int_E \log |h(x, y)| \, d\mu_\lambda(y)$$

with the following properties:

$$u_\lambda(x)\begin{cases}\leqslant \dfrac{\gamma_\lambda}{\lambda}+f(x) & \text{almost everywhere on } E,\\ = \dfrac{\gamma_\lambda}{\lambda}+f(x) & \text{almost everywhere on } E_\lambda,\end{cases}$$

$$\gamma_\lambda = \text{const} = \int_E\int_E \log|h(x,y)|\,d\mu_\lambda d\mu_\lambda - \lambda\int_E f(x)\,d\mu_\lambda,$$

where E_λ is the support of the mass distribution defined by extremal points on E.

6. The sequent results [9] concern the Tchebycheff interpolation polynomials.

We introduce the following notation:

$f(x)$ is semicontinuous and bounded function on E;

$$w(z, c^{(n)}, f) = |z-c_1|\ldots|z-c_n|\exp\bigl(-nf(z)\bigr);$$

$$\tau_n(E,f) = \max_{z\in E} w(z, \xi^{(n)}, f);$$

$$\varrho_n(E,f) = \min_{c^{(n)}\in E}\{\max_{z\in E} w(z, c^{(n)}, f)\} = \max_{z\in E} w(z, \eta^{(n)}, f).$$

Then

$$T_n(z,E,f) = w(z,\xi^{(n)},f)\exp nf(z),$$

$$P_n(z,E,f) = w(z,\eta^{(n)},f)\exp nf(z),$$

$$\lim_{n\to\infty}\sqrt[n]{|T_n(z,E,f)|} = \lim_{n\to\infty}\sqrt[n]{|P_n(z,E,f)|} \quad \text{for} \quad z\notin E,$$

$$\lim_{n\to\infty}\sqrt[n]{\tau_n(E,f)} = \lim_{n\to\infty}\sqrt[n]{\varrho_n(E,f)} = 1/\lim_{z\to\infty}\frac{\Phi(z,f)}{|z|}.$$

7. A very short proof of the uniqueness of the equilibrum mass distribution on a given compact E has been given by Bach [1].

8. *Coefficient problem.* Let

$$\text{(i)}\qquad w = f(z) = z + a_2z^2 + a_3z^3 + \ldots$$

be an analytic univalent function in the unit circle K: $|z|<1$ and let Δ be the image of K by (i). We denote by D the image of Δ under the transformation $\xi = 1/w$. D is a bounded simply connected domain which contains the point ∞. The boundary E of D is a bounded continuum with capacity 1. Let $\eta^{(n)}$ be an n-th extremal system of points on E, i.e. a system $\eta_1,\ldots,\eta_n$ such that for any $\xi_1,\ldots,\xi_n\in E$ we have

$$\prod_{j<k}|\eta_j-\eta_k| \geqslant \prod_{j<k}|\xi_j-\xi_k|.$$

It has been proved in [8] that

1. the limits

$$\lim_{n\to\infty}\frac{\eta_1^k+\eta_2^k+\ldots+\eta_n^k}{n} = s_k, \qquad k=1,2,\ldots,$$

exist,

2. the coefficients b_k of the inverse function

$$z = w + b_2w^2 + b_3w^3 + \ldots$$

are given by

$$b_{k+1} = \frac{1}{k}(s_k + b_2s_{k-1} + \ldots + b_ks_1),$$

3. the coefficients a_2, a_3 and a_4 of the function $f(z)$ are given by

$$a_2 = -s_1, \qquad a_3 = \frac{3s_1^2-s_2}{2}, \qquad a_4 = \frac{-8s_1^3+6s_1s_2-s_3}{3}.$$

Further coefficients a_k can be easily calculated from the identity

$$z = (z+a_2z^2+\ldots) + b_2(z+a_2z^2+\ldots)^2 + \ldots$$

Moreover, the following results have been obtained:

(I) If $|a_3| = \max$, then $|b_3| = \max$;

(II) the sharp inequality $|s_2|\leqslant 6$;

(III) $|a_4+b_4|\leqslant 10$;

(IV) if $|a_4| = \max$, then $|b_4| = 14$.

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EXTREMAL POINTS IN THE SPACE C^n

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This is a report* on an extension of Leja's method to problems in several complex variables.

1. Interpolation formulas. The Lagrange interpolation formulas for ordinary polynomials of n variables and for homogeneous polynomials of n variables are basic tools in the method of extremal points in the space C^n of n complex variables.

Any polynomial $P_\nu(z)$ of degree ν may be written in the form

$$P_\nu(z) = \sum_{l=1}^{\nu_*} a_{k_{1l}k_{2l}\ldots k_{nl}} z_1^{k_{1l}} z_2^{k_{2l}} \ldots z_n^{k_{nl}}, \tag{1.1}$$

where $(k_{1l}, k_{2l}, \ldots, k_{nl})$, $l = 1, 2, \ldots, \nu_*$, $\nu_* = \binom{\nu+n}{n}$, is the sequence of all solutions in non-negative integers of the inequality $k_1 + k_2 + \ldots + k_n \leqslant \nu$.

Analogously, any homogeneous polynomial $Q_\nu(z)$ of degree ν may be written in the form

$$Q_\nu(z) = \sum_{l=1}^{\nu_0} a_{h_{1l}h_{2l}\ldots h_{nl}} z_1^{h_{1l}} z_2^{h_{2l}} \ldots z_n^{h_{nl}}, \tag{1.2}$$

where $(h_{1l}, h_{2l}, \ldots, h_{nl})$, $l = 1, 2, \ldots, \nu_0$, $\nu_0 = \binom{\nu+n-1}{n-1}$, is a complete sequence of the solutions in non-negative integers of the equation $h_1 + h_2 + \ldots + h_n = \nu$.

Suppose $p^{(\nu)} = (p_1, p_2, \ldots, p_{\nu_*})$ is a system of ν_* points $p_i = (z_{1i}, \ldots, z_{ni})$, $i = 1, 2, \ldots, \nu_*$, of C^n such that the determinant $V(p^{(\nu)}) = V(p_1, \ldots, p_{\nu_*})$ defined by

$$V(p^{(\nu)}) = \det[z_{1i}^{k_{1l}} z_{2i}^{k_{2l}} \ldots z_{ni}^{k_{nl}}], \quad i, l = 1, 2, \ldots, \nu_*, \tag{1.3}$$

is different from zero. Then the following interpolation formula holds:

$$P_\nu(z) = \sum_{i=1}^{\nu_*} P_\nu(p_i) L^{(i)}(z, p^{(\nu)}), \quad z \in C^n, \tag{1.4}$$

* Presented to the Third Conference on Analytic Functions, held in Cracow, 30. VIII-4. IX. 1962.