COLLOQUIUM MATHEMATICUM

VOL. XI 1964 FASC. 2

ON SOME RECENT RESULTS IN THE THEORY OF ANALYTIC FUNCTIONS

BY

J. KRZYŻ (LUBLIN)

This is a report * on some results in the theory of analytic functions obtained in Lublin in 1959-1962. Since the problems concerning the relations between majorization and subordination are treated in a separate paper by Bielecki [1], the corresponding results will not be mentioned here. The results dealt with here concern mainly

(i) the class S of functions univalent in the unit circle K with standard normalization and various subclasses of S;

(ii) the class M of functions univalent in K with Montel's normalization $F(0) = 0$, $F(z_0) = 1$ $(0 < |z_0| < 1)$.

Lewandowski [13, 14] showed that the class L of close-to-convex functions introduced in 1952 by Kaplan [5] is identical with the class of "linearly accessible functions" introduced by Biernacki [4] in 1936. Since the class L contains such classes of functions as starlike with respect to an arbitrary point, convex, convex in one direction, with derivative of positive real part, Lewandowski's result arose great interest in this class in the Lublin centre. The original proof of Lewandowski presented at the Conference on Analytic Functions in Lublin four years ago, was rather involved. Recently Lewandowski published together with Bielecki [2] a new, concise and elegant proof based on the concept of homotopy.

Using some results of Biernacki, Złotkiewicz [18] found recently a sharp estimation of $\arg zf'(z)/f(z)$ and $\arg f(z)/z$ for $f \in L$. Although Biernacki [4] determined the domains of variability of $zf'(z)/f(z)$ and $f(z)/z$, for a fixed z and f ranging over L, the results were stated by him in such a form that explicite bounds of argument could not be derived immediately from his final statements.

In connection with a result of A. Marx (for reference see [12], p. 26) concerning the domain of variability of $f(z)/z$ for a fixed $z \in K$ and f ran-

* Presented to the Third Conference on Analytic Functions held in Cracow 30. VIII - 4. IX. 1962.

ging over the class S^* of functions starlike with respect to the origin, Krzyż [12] determined the domain of variability D_r of $W = \log f'(re^{i\theta})$ for $f \in L$ and fixed r, θ; D_r does not depend on θ, is symmetric with respect to the lines $\operatorname{Re} W = -\log(1-r^2)$, $\operatorname{Im} W = 0$ and arises by reflecting a convex arc (whose equation can be given in an explicit form) with respect to the axes of symmetry. In particular, $|\arg f'(re^{i\theta})| \leqslant 4 \arcsin r$. This implies e. g. that for $|z| < \varrho_0$, where $\varrho_0 = \sin\frac{1}{8}\pi$, we have $\operatorname{Re} f' > 0$ for any $f \in L$.

A similar problem was treated by Lewandowski [14] who found the radius of starshapedness $r_L^* = 4\sqrt{2}-5 = 0{,}6568\ldots$ for the class L which shows to be just slightly greater than the corresponding radius for the whole class S.

In 1952 Kaplan [5] put the problem of determination of the radius of close-to-convexity r_0 for the class S. He remarked that, if the map $f(K(r))$ of a circle $K(r) = (z\colon |z| < r)$ under f is a close-to-convex domain, so is $f(K(r_1))$ for any $r_1 < r$. Thus the problem arises to determine the greatest number r_0 such that for any $f \in S$ and any r, $0 < r < r_0$, the domain $f(K(r))$ is close-to-convex. Clearly $r_0 > r^* = \tanh \pi/4$, since the starshapedness implies close-to-convexity. Recently Krzyż [8] could solve this problem; r_0 turns out to be the unique root of a transcendental equation $\varphi(r) = 0$, which is rather complicated, but in view of the fact that the l. h. s. $\varphi(r)$ is a monotonic function of r, it can be solved easily by successive substitutions. We have $0{,}80 < r_0 < 0{,}81$.

Four years ago Lewandowski [15] found a generalization of the Marx's result mentioned above. He found the domain of variability of the functional $[z_0 f(z)/z f(z_0)]^{1/2}$ for fixed z and z_0, and f ranging over S^*. The domain of variability turns out to be the circle with radius $|z-z_0|/(1-|z|^2)$ and centre $(1-\bar{z}z_0)/(1-|z|^2)$. Using this he also determined (see [16]) the domain of variability of $f(z_1)/f(z_2)$ for z_1 and z_2 moving on the circumference $|z| = r$ and f ranging over S^*. This domain (depending on r) is of considerable importance in the theory of subordination.

Also a result of Złotkiewicz [18] is connected with the class S^*. Using a result of G. M. Golusin he found a simple derivation of Hummel's variational formula for $f \in S^*$.

Some four years ago Z. Charzyński put the problem of evaluating the local and global distortions for bounded, convex mappings of the unit circle. By means of Hadamard's variational formulae Krzyż [6, 7] found precise bounds for $|f(z)|$, $|f''(0)|$ and also for $(1-|z|^2)|f'(z)|$, the last estimation depends however on $|f|$. The extremal function is the same in all cases and realizes the conformal mapping of the unit circle on a circular segment of radius M the segment being chosen so that the inner conformal radius of it with respect to the centre is equal to 1.

Bielecki, Krzyż and Lewandowski [3] determined the exact domain of variability of $f(z)$, if z is a fixed complex number contained in the unit disc, and f is ranging over the class of typically-real functions with a preassigned second Taylor coefficient at the origin. For real z the domain degenerates to a segment determined previously by J. A. Jenkins.

Zmorovič [20] who was concerned in detail with functions f such that $\operatorname{Re} f' > 0$ put the question, if this class would be a subclass of S^*. Both classes are evidently subclasses of L. As pointed out by Krzyż [10] neither of both classes does contain the other.

In 1933 P. Montel put the problem of estimation of $|F(z)|$ for functions univalent in K which satisfy $F(0) = 0$, $F(z_0) = 1$, $0 < |z_0| < 1$. Let M denote the class of functions subject to such a normalization. Krzyż solved Montel's problem by using Schiffer's variational method. This, as well as an approximate solution in terms of elementary functions, is contained in [9] and [10].

Lewandowski [15] found in an elementary way the Koebe constant for the class M, i. e. the radius of the greatest disc covered by the values of all functions $F \in M$; it is equal to $(1-|z_0|^2)/4|z_0|$.

It is well known that, if $\sum_{n=2}^{\infty} n|a_n| \leqslant |a_1|$, then the function $a_1 z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in the unit circle. Mrs. Piłat [17] obtained some theorems on the class of functions which are subject, moreover. to the Montel normalization.

REFERENCES

[1] A. Bielecki, *Quelques résultats récents sur les majorantes dans la théorie des fonctions holomorphes*, Colloquium Mathematicum 11 (1964), p. 141-145.

[2] A. Bielecki et Z. Lewandowski, *Sur un théorème concernant les fonctions univalentes de M. Biernacki*, Annales Polonici Mathematici 12 (1962), p. 61-63.

[3] A. Bielecki, J. Krzyż and Z. Lewandowski, *On typically-real functions with a preassigned second coefficient*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 10 (1962), p. 205-208.

[4] M. Biernacki, *Sur la représentation conforme des domaines linéairement accessibles*, Prace Matematyczno-Fizyczne 44 (1936), p. 293-314.

[5] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Mathematical Journal 1 (1952), p. 169-185.

[6] J. Krzyż, *Distortion theorems for bounded convex functions*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 8 (1960), p. 625-627.

[7] — *Distortion theorems for bounded convex functions II*, Annales Universitatis Mariae Curie-Skłodowska, Sectio A, 14 (1960), p. 5-16.

[8] — *The radius of close-to-convexity within the family of univalent functions*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 10 (1962), p. 201-204.

[9] — *On a problem of P. Montel*, Annales Polonici Mathematici 12 (1962), p. 55-60.

[10] — *Some remarks concerning my paper: On univalent functions with two preassigned values*, Annales Universitatis Mariae Curie-Skłodowska, Sectio A, 16 (1962), p. 129-136.

[11] — *A counterexample concerning univalent functions*, Folia Societatis Scientiarum Lubliniensis 2 (1962), p. 57-58.

[12] — *Some remarks on close-to-convex functions*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 12 (1964), p. 25-28.

[13] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes I*, Annales Universitatis Mariae Curie-Skłodowska, Sectio A, 12 (1958), p. 131-146.

[14] — *Sur l'identité de certaines classes de fonctions univalentes II*, ibidem 14 (1960), p. 19-46.

[15] — *Sur certaines classes de fonctions univalentes dans le cercle-unité*, ibidem 13 (1959), p. 115-126.

[16] — *Starlike majorants and subordination*, ibidem 15 (1961), p. 79-84.

[17] B. Piłat, *On a class of univalent functions*, Folia Societatis Scientiarum Lubliniensis 2 (1962), p. 62-63.

[18] E. Złotkiewicz, *On a variational formula for starlike functions*, Annales Universitatis Mariae Curie-Skłodowska, Sectio A, 15 (1961), p. 111-113.

[19] — *On the precise bounds of* $\arg f(z)/z$ *and* $\arg zf'(z)/f(z)$ *for close-to-convex functions*, Folia Societatis Scientiarum Lubliniensis 3 (1963), to appear.

[20] В. А. Зморович, *Про деякі задачі теорії унівалентних функцій*, Наукові записки Київського Державного Університету им. Шевченка 11 (1952), p. 83-94 (in Ukrainian).

Reçu par la Rédaction le 15. 2. 1963

COLLOQUIUM MATHEMATICUM

VOL. XI 1964 FASC. 2

SOME APPLICATIONS OF THE METHOD OF EXTREMAL POINTS IN THE THEORY OF ANALYTIC FUNCTIONS OF ONE COMPLEX VARIABLE

BY

J. GÓRSKI (CRACOW)

We present here an outline* of some recent results obtained with the aid of Leja's method.

Let R be a topological space, E a bounded closed set and $\omega(x, y)$ a real function of two points $x, y \in R$ which satisfies the following conditions:

$$\omega(x, y) \geqslant 0, \quad \omega(x, y) = \omega(y, x).$$

Denote by $f(x)$ a real function defined on E and consider the product

$$V(p^{(n)}, \omega, \lambda f) \doteq \prod_{1 \leqslant i < k \leqslant n} \omega(p_i, p_k) \exp \lambda [f(p_i) + f(p_k)] \tag{1}$$

where $\lambda > 0$ is a parameter and $p^{(n)} = (p_1, p_2, \ldots, p_n)$ an arbitrary system of n points of E.

Let $V_n(\omega, \lambda f)$ be the upper bound of the product (1) when the system $p^{(n)}$ varies in E. When $f(x)$ and $\omega(x, y)$ are continuous functions there exists at least one system of n points $q^{(n)} \epsilon E$ such that

$$V_n(\omega, \lambda f) = V(q^{(n)}, \omega, \lambda f).$$

A system $q^{(n)} = (q_1, \ldots, q_n)$ is called the *n-th extremal system* of points of E with respect to $\omega(x, y)$ and $\lambda f(x)$.

It has been proved by Leja [6] that the limit

$$\lim_{n \to \infty} V_n(\omega, \lambda f)^{2/n(n-1)} = v(\omega, \lambda f)$$

exists. The number $v(\omega, \lambda f) \geqslant 0$ is called the *ecart* of the set E.

Let $\Phi^{(j)}(x, p^{(n)}, \omega, \lambda f)$, $j = 1, \ldots, n$, be the sequence of functions

$$\Phi^{(j)}(x, p^{(n)}, \omega, \lambda f) = \left[\prod_{\substack{k \neq j \\ k=1}}^{n} \frac{\omega(x, p_k)}{\omega(p_j, p_k)} \right] \exp n\lambda f(p_j).$$

* Presented to the Third Conference on Analytic Functions, held in Cracow, 30. VIII-4. IX. 1962.