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SOME APPROXIMATE INTEGRATION FORMULAS OF STATISTICAL INTEREST

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1. Introduction. A series of papers (Mikulski, Rudzki and Wiśniewski [3], Oderfeld [4], Rudzki [5], [7], [8]) discuss the problem of improving the accuracy of estimation of the mean value of a numerical characteristic of a shapeless product as compared with random sampling by the use of a suitable constant sampling pattern or of a suitable approximate integration formula chosen on the basis of informations we have about the distribution of the characteristic within the geometrical body occupied by the product. Put another way, the question is what is the best way of locating a given number of measurement points in order that the least possible error of estimation be achieved. Let us mention three cases pertinent to the description of the distribution of the characteristic within the body of the product by mathematical assumptions.

Case I. The main subject of discussion was the case we propose to call three-dimensional. In this case it is assumed that the body of the product is convex and that the value w of the characteristic in the point P of the body depends only on the ratio r of the lengths of the segments PP' and SP', where S is a known distinguished point of the body (usually its geometrical centre or the geometrical centre of the plane base of the body) and P' is a point of intersection with the surface of the body of a half-line directed from S to P (see fig. 1.1). So we have the relation

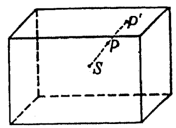

Fig. 1.1

$$w = f(r), \quad 0 \leqslant r \leqslant 1, \tag{1.1}$$

and the mean value $\overline{w}$ of w over the body of the product is given by

$$\overline{w} = \int_0^1 3r^2 f(r)\,dr. \tag{1.2}$$

A bale of cotton or of tobacco leaves, or a bag of corn, with humidity as the considered characteristic, may serve as examples.

Case II. The simplest situation pertinent to the two-dimensional case may be described as follows.

The body of the product is a circular cone. Its axis s is a distinguished line and the value w of a characteristic in a point P depends on the ratio r of the lengths of the segments PP' and SP', where S is the point of intersection with s of a straight line passing through P and parallel to a generating line of the cone, while P' is the point of intersection of this line with the base of the cone (see fig. 1.2). In this case we have for the mean value $\overline{w}$ of the characteristic the expression

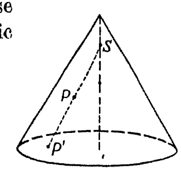

Fig. 1.2

(1.3) $$\overline{w} = \int_0^1 2rf(r)\,dr,$$

where again $w = f(r)$ for $0 \leqslant r \leqslant 1$.

The stratification of loose material arising when it is poured on a cone-shaped heap by a conveyer may serve as an example.

Case III. In this one-dimensional case it is assumed that the body of product may be conceived as arisen by a movement perpendicular to a given plane σ of a figure lying originally on σ and then remaining parallel to it, and that the value w of a characteristic in a point P of the body depends on the ratio r of the lengths of the segments PP' and SP', where S is a point of intersection with σ of a straight line perpendicular to σ passing through P, and P' is the other point of intersection of this line with the surface of the body (see fig. 1.3). In this case we have

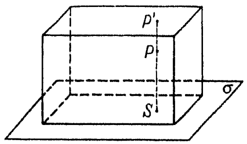

Fig. 1.3

(1.4) $$\overline{w} = \int_0^1 f(r)\,dr,$$

where, as before, $w = f(r)$ for $0 \leqslant r \leqslant 1$.

A railway truck loaded with loose material may serve as an example.

A number of problems concerning the estimation of $\overline{w}$ by a linear combination $c_1 f(r_1) + \ldots + c_n f(r_n)$, where for a given n either the c's or the r's or both are to be chosen so as to minimize the supremum of the absolute difference

$$\left|\overline{w} - (c_1 f(r_1) + \ldots + c_n f(r_n))\right|$$

for certain classes of functions $f(r)$, have been discussed.

So in [3] and [4] some simple choices of $f(r)$, admitting an exact evaluation of $\overline{w}$, have been considered. In [4] and [7] a class of monotonic functions $f(r)$ such that

$$1 - r^b \leqslant f(r) \leqslant 1 - r^a, \quad 1 \leqslant a < b,$$

in [7] with the additional condition

$$\int_0^1 (1-r^a)\,dr - \int_0^1 (1-r^b)\,dr = k,$$

has been discussed. In [5] the class of functions

$$f(r) = 1 - r^m, \quad m \geqslant 1,$$

and in [8] the class of functions

$$f(r) = 1 - r^m, \quad 0 \leqslant m \leqslant 1,$$

have been subject of interest.

However, optimal systems of nodes r_i and of weights c_i have been got only for a few small values of n.

Problems of determinig the c's and r's in the linear form $c_1 f(r_1) + \ldots + c_n f(r_n)$ so as to minimize

$$\sup_{f \in H} \left| \int_0^1 f(r)\,dr - (c_1 f(r_1) + \ldots + c_n f(r_n)) \right|$$

for a given class H of functions $f(r)$, $0 \leqslant r \leqslant 1$, have been considered by other authors (see [9]-[11]). For H have been taken classes of functions which have certain regularity properties, Lipschitz condition or bounded variation of $f(r)$ being instances of most general conditions.

Our aim is to prove two simple results of this sort under assumptions which seem to be suggested by the examples of this section: the first for bounded non-decreasing functions (Theorem 1), the other for bounded non-decreasing functions which independently of this fulfil a Lipschitz condition (Theorem 2).

Theorem 1 is closely related to a result by Steinhaus and Trybuła [2] (the reader may see also [1] and [6]). There the question was how to divide the segment $0 \leqslant r \leqslant 1$ into n disjoint parts $S_1, \ldots, S_n$ and to attach a number r_i to each S_i so that $\sup_{0 \leqslant r \leqslant 1} |r - r'|$, where $r' = r_i$ if $r \in S_i$, be a minimum. We can restate it as follows: what is a minimax linear approximate integration formula with $n-1$ nodes for a class of non-increasing functions $f(r)$, $0 \leqslant r \leqslant 1$, taking on values 1 and 0 only? The answer is that we should divide the segment $0 \leqslant r \leqslant 1$ into equal parts $S_i = \{r\colon (i-1)/n < r \leqslant i/n\}$ with $0 \in S_1$ and choose $r_i = (2i-1)/2n$, $i = 1, 2, \ldots, n$, which amounts to the same as to take the points $1/n$, $2/n, \ldots, (n-1)/n$ for nodes and to use

$$\frac{1}{2n}+\frac{1}{n}\left(f\left(\frac{1}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)\right)$$

as the approximate integration formula.

Theorem 1 may be regarded as a kind of limiting form of theorem 2, which arises when the constant in Lipschitz condition increases indefinitely. Thus in view of theorem 2 its proof is in a sense superfluous. However, we bring the proof because of its simplicity.

Finally, in section 3, we reword our theorems so as to get minimax approximate integration formulas relating to the integration with respect to any normed absolutely continuous measure.

2. Approximate integration formulas. In this section we shall formulate and prove two theorems on approximate integration of functions $f(x)$ defined for $0 \leqslant x \leqslant 1$. We shall give them the form of statements about estimation games between nature and the statistician.

Let H be the class of non-decreasing functions $f(x)$ defined for $0 \leqslant x \leqslant 1$ and with values in the interval $0 \leqslant f(x) \leqslant K$ which are continuous at $x = 0$ and $x = 1$. Let B be the set of vectors $(\boldsymbol{c}, \boldsymbol{x}) = (c_1, \ldots, c_n; x_1, \ldots, x_n)$ such that $c_1+\ldots+c_n = 1$ and $0 < x_1 < \ldots < x_n < 1$. The numbers $c_1, \ldots, c_n$ will be called *weights*, the numbers $x_1, \ldots, x_n$ will be called *nodes*.

THEOREM 1. *Consider an estimation game between nature, which chooses f from H, and statistician, who chooses $(\boldsymbol{c}, \boldsymbol{x})$ from B. Let*

$$r(f,(\boldsymbol{c},\boldsymbol{x})) = \left|\int_0^1 f(x)\,dx-(c_1 f(x_1)+\ldots+c_n f(x_n))\right| \tag{2.1}$$

be the payoff in this game. Then the unique minimax strategy of the statistician is the vector $(\boldsymbol{c}_0, \boldsymbol{x}_0)$ defined by the equalities

$$\boldsymbol{c}_0 = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right), \qquad \boldsymbol{x}_0 = \left(\frac{1}{2n}, \frac{3}{2n}, \ldots, \frac{2n-1}{2n}\right), \tag{2.2}$$

and the minimax risk of the statistician is given by

$$\inf_{(\boldsymbol{c},\boldsymbol{x})\in B}\ \sup_{f\in H} r(f,(\boldsymbol{c},\boldsymbol{x})) = \frac{K}{2n}. \tag{2.3}$$

The proof of theorem 1 will be based on three lemmas.

LEMMA 1. *For a given $(\boldsymbol{c}, \boldsymbol{x}) \in B$ let $H_{(\boldsymbol{c},\boldsymbol{x})}$ be the class of functions $f \in H$ which are constant in the intervals $0 < x < x_1$, $x_1 < x < x_2, \ldots, x_{n-1} < x < x_n$, $x_n < x < 1$. For every $(\boldsymbol{c}, \boldsymbol{x}) \in B$ there is a function $g \in H_{(\boldsymbol{c},\boldsymbol{x})}$ such that*

$$\sup_{f\in H} r(f,(\boldsymbol{c},\boldsymbol{x})) = r(g,(\boldsymbol{c},\boldsymbol{x})). \tag{2.4}$$

Proof. Let us fix $(\boldsymbol{c}, \boldsymbol{x}) \in B$. We shall confine ourselves to proving that for every function $f \in H$ there is a function $g \in H_{(\boldsymbol{c},\boldsymbol{x})}$ such that

$$r(f,(\boldsymbol{c},\boldsymbol{x})) \leqslant r(g,(\boldsymbol{c},\boldsymbol{x})). \tag{2.5}$$

Indeed (see fig. 2.1), if

$$\int_0^1 f(x)\,dx \leqslant c_1 f(x_1)+\ldots+c_n f(x_n), \tag{2.6}$$

it is sufficient to take for g a function continuous on the right and fulfilling the equalities

$$g(x) = \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)\,dx \tag{2.7}$$

for $x_i < x < x_{i+1}$, $x_0 = 0$, $x_{n+1} = 1$ $(i = 0, 1, \ldots, n)$.

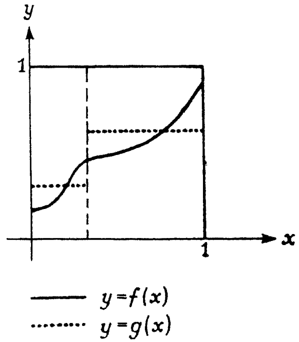

Fig. 2.1

If inequality (2.6) does not hold, a function g fulfilling equations (2.7) which is continuous on the left will serve the purpose.

LEMMA 2. *If $(\boldsymbol{c}_0, \boldsymbol{x}_0)$ is defined by (2.2), we have*

$$\sup_{f\in H} r(f,(\boldsymbol{c}_0,\boldsymbol{x}_0)) = \frac{K}{2n}. \tag{2.8}$$

In view of lemma 1 we may restrict ourselves to consider only functions $f \in H_{(\boldsymbol{c}_0,\boldsymbol{x}_0)}$. For a function $f \in H_{(\boldsymbol{c}_0,\boldsymbol{x}_0)}$ with fixed values in the intervals $0 < x < x_1, \ldots, x_n < x < 1$ the values $f(x_i)$, $i = 1, 2, \ldots, n$, are maximum, when the function f is continuous on the right (see fig. 2.2). We have then

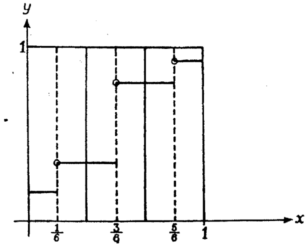

Fig. 2.2

$$\frac{1}{n}f(x_i) = \int_{x_i}^{x_{i+1}} f(x)\,dx, \quad i = 1, 2, \ldots, n-1,$$

and thus

$$r(f,(\boldsymbol{c}_0,\boldsymbol{x}_0)) = \left|\frac{1}{n}f(x_n)-\int_0^{1/2n} f(x)\,dx-\int_{(2n-1)/2n}^{1} f(x)\,dx\right|$$
$$= \left|\frac{1}{n}f(x_n)-\frac{1}{2n}f(0)-\frac{1}{2n}f(x_n)\right| = \frac{1}{2n}|f(x_n)-f(0)| \leqslant \frac{K}{2n}$$

with the equality holding if $f(0) = 0$ and $f(1) = K$.

Similarly we get $r(f, (\mathbf{c}_0, \boldsymbol{x}_0)) \leqslant K/2n$ for functions $f \epsilon H_{(\mathbf{c}_0, \boldsymbol{x}_0)}$ which are continuous on the left, and thus this inequality holds for every $f \epsilon H_{(\mathbf{c}_0, \boldsymbol{x}_0)}$.

LEMMA 3. *If* $(\mathbf{c}, \boldsymbol{x}) \neq (\mathbf{c}_0, \boldsymbol{x}_0)$, *where* $(\mathbf{c}_0, \boldsymbol{x}_0)$ *is given by* (2.2), *then*

$$\sup_{f\epsilon H} r(f, (\mathbf{c}, \boldsymbol{x})) > \frac{K}{2n}.$$

Proof. We shall prove that if the assumptions of lemma 3 hold, then there is a function $f \epsilon H$ such that $r(f, (\mathbf{c}, \boldsymbol{x})) > K/2n$.

For a given $(\mathbf{c}, \boldsymbol{x}) \epsilon B$ define a function $s_{(\mathbf{c}, \boldsymbol{x})}(x) = s(x)$ by

$$s(x) = \begin{cases} -x & \text{for} \quad 0 \leqslant x \leqslant x_1, \\ -x + \sum_{i: x_i < x} c_i & \text{for} \quad x_1 < x \leqslant 1. \end{cases} \tag{2.9}$$

The function $s(x)$ has thus jumps $c_1, \ldots, c_n$ at points $x_1, \ldots, x_n$, and otherwise it has a derivative equal to -1. Moreover, because of the condition $c_1 + \ldots + c_n = 1$ we may put by continuity $s(0) = s(1) = 0$ (see fig. 2.3). Let

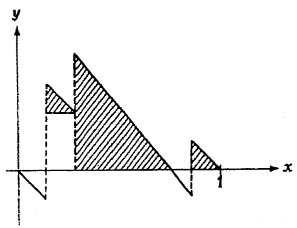

Fig. 2.3

$$C = \sup_{0 \leqslant x \leqslant 1} |s(x)| = \max\{-s(x_1 - 0), s(x_1 + 0), \ldots, -s(x_n - 0), s(x_n + 0)\},$$

where $s(x-0)$ is the left-hand limit and $s(x+0)$ is the right-hand limit of $s(x)$ at x. Suppose we have $C = -s(x_i - 0)$. If we take for f a function which is continuous on the right and fulfils the equalities

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 < x < x_i, \\ K & \text{for} \quad x_i < x < 1, \end{cases}$$

we shall have

$$\int_0^1 f(x)\,dx = K(1 - x_i), \qquad c_1 f(x_1) + \ldots + c_n f(x_n) = K(c_i + \ldots + c_n),$$

and thus

$$\begin{aligned} r(f, (\mathbf{c}, \boldsymbol{x})) &= |K(1 - x_i) - K(1 - c_1 - \ldots - c_{i-1})| \\ &= K|(c_1 + \ldots + c_{i-1}) - x_i| = KC. \end{aligned}$$

If we have $C = s(x_i + 0)$ for some i, the construction of a function f for which $r(f, (\mathbf{c}, \boldsymbol{x})) = KC$ is analogous. Thus we see that

$$\sup_{f\epsilon H} r(f, (\mathbf{c}, \boldsymbol{x})) \geqslant KC.$$

Now if $(\mathbf{c}, \boldsymbol{x}) \neq (\mathbf{c}_0, \boldsymbol{x}_0)$, then $C > 1/2n$. To see this, note that we have $c_i = s(x_i + 0) - s(x_i - 0)$ and $c_1 + \ldots + c_n = 1$, which implies that $C \geqslant \frac{1}{2}\max\{|c_1|, \ldots, |c_n|\} \geqslant 1/2n$, and the equation $C = 1/2n$ is attained if and only if all c's equal $1/n$ and all sums $s(x_i + 0) + s(x_i - 0)$ equal 0. Then, however, $(\mathbf{c}, \boldsymbol{x}) = (\mathbf{c}_0, \boldsymbol{x}_0)$.

Lemma 3 is thus proved.

Theorem 1 is an immediate consequence of lemmas 1, 2, and 3.

Remark 1. If we replace in theorem 1 the condition $0 \leqslant f(x) \leqslant K$ by $f(1) \leqslant f(0) + K$, we can drop the condition $c_1 + \ldots + c_n = 1$.

Remark 2. The assertion of theorem 1 remains unchanged, if we drop the condition $c_1 + \ldots + c_n = 1$ and replace the class H by the class of functions $f(x)$ with variation $Vf \leqslant K$.

Examples of section 1 seem to suggest that it would be interesting to narrow the class H of functions f by imposing on them in addition a condition of differentiability and of boundedness of the derivative. In order to avoid inessential discussions in the proof and to get clearer formulations we shall choose, instead, a Lipschitz condition.

Let thus L be the class of non-decreasing functions $f(x)$ defined for $0 \leqslant x \leqslant 1$, with values in the interval $0 \leqslant f(x) \leqslant K$ and satisfying the Lipschitz condition

$$|f(x') - f(x'')| \leqslant M|x' - x''|.$$

THEOREM 2. *Consider an estimation game between nature, which chooses a function f from L, and a statistician, who chooses a vector $(\mathbf{c}, \boldsymbol{x})$ of weights and nodes from B. Let the payoff be defined by* (2.1). *In this game the unique minimax strategy of the statistician is the vector $(\mathbf{c}_0, \boldsymbol{x}_0)$ defined by* (2.2), *and the minimax risk of the statistician is given by*

$$\inf_{(\mathbf{c}, \boldsymbol{x})\epsilon B} \sup_{f\epsilon L} r(f, (\mathbf{c}, \boldsymbol{x})) = \begin{cases} \dfrac{1}{2n}\,\dfrac{M}{4} & \text{if} \quad M \leqslant 2K, \\ \dfrac{K}{2n}\left(1 - \dfrac{K}{M}\right) & \text{if} \quad 2K \leqslant M. \end{cases} \tag{2.10}$$

We shall base the proof of theorem 2 on four lemmas.

LEMMA 4. *For every $(\mathbf{c}, \boldsymbol{x}) \epsilon B$ we have the relation*

$$\sup_{f\epsilon L} r(f, (\mathbf{c}, \boldsymbol{x})) = \sup_{f\epsilon L_0} r(f, (\mathbf{c}, \boldsymbol{x})), \tag{2.11}$$

where L_0 is the class of functions $f \in L$ which have a piecewise constant derivative.

Proof. Let us fix $(c, x) \in B$. It is easily seen that for every function $f \in L$ there is a function $g \in L_0$ such that $f(x_i) = g(x_i)$, $i = 1, 2, \ldots, n$, and that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

Thus lemma 4 follows.

Lemma 5 (compare Nikolskiĭ [9], p. 24-26). *Given $(c, x) \in B$, define a function $s_{(c,x)}(x) = s(x)$ by (2.9). Then we have for every $f \in L_0$ and every $(c, x) \in B$ the relation*

$$\int_0^1 f(x)\,dx - \big(c_1 f(x_1) + \ldots + c_n f(x_n)\big) = \int_0^1 s(u) f'(u)\,du. \tag{2.12}$$

Proof. Without loss of generality we can assume that $f(0) = 0$. Then we can write

$$f(x) = \int_0^x f'(u)\,du,$$

or, with the use of the auxiliary function

$$S(u) = \begin{cases} 1 & \text{for} \quad u \geqslant 0, \\ 0 & \text{for} \quad u < 0, \end{cases}$$

we have

$$f(x) = \int_0^1 S(x-u) f'(u)\,du.$$

Owing to this we have

$$\int_0^1 f(x)\,dx - \big(c_1 f(x_1) + \ldots + c_n f(x_n)\big)$$

$$= \int_0^1 \Big\{ \int_0^1 S(x-u) f'(u)\,du - \sum_{i=1}^n c_i \int_0^1 S(x_i - u) f'(u)\,du \Big\}\,dx$$

$$= \int_0^1 \Big\{ \int_0^1 S(x-u)\,dx - \sum_{i=1}^n c_i S(x_i - u) \Big\} f'(u)\,du$$

$$= \int_0^1 \Big\{ 1 - u - \sum_{i: x_i \geqslant u} c_i \Big\} f'(u)\,du = \int_0^1 \Big\{ 1 - u - \Big(1 - \sum_{i: x_i < u} c_i\Big)\Big\} f'(u)\,du$$

$$= \int_0^1 \Big\{ -u + \sum_{i: x_i < u} c_i \Big\} f'(u)\,du = \int_0^1 s(u) f'(u)\,du.$$

Lemma 6. *If (c_0, x_0) is given by (2.2), we have*

$$\sup_{f \in L_0} r\big(f, (c_0, x_0)\big) = \begin{cases} \dfrac{1}{2n}\,\dfrac{M}{4} & \text{for} \quad M \leqslant 2K, \\[2mm] \dfrac{K}{2n}\Big(1 - \dfrac{K}{M}\Big) & \text{for} \quad 2K \leqslant M. \end{cases} \tag{2.13}$$

Lemma 7. *If $(c, x) \neq (c_0, x_0)$, where (c_0, x_0) is given by (2.2), then*

$$\sup_{f \in L_0} r\big(f, (c, x)\big) > \sup_{f \in L_0} r\big(f, (c_0, x_0)\big).$$

Proof. In view of lemma 5 and the form of function $s_{(c_0, x_0)}(x) = s_0(x)$ we see that $r\big(f, (c_0, x_0)\big)$ is maximum if $f(0) = 0$, and

$$f'(x) = \begin{cases} M & \text{if} \quad s_0(x) > a_0, \\ 0 & \text{if} \quad s_0(x) < a_0, \end{cases}$$

where a_0 is the least non-negative a for which

$$\operatorname{mes}\{u: s_0(u) > a\} \leqslant \frac{K}{M}.$$

There remains a simple computation which leads to (2.13) and thus proves lemma 6.

To prove lemma 7 it is sufficient to verify that under assumptions of lemma 7 there is a set T of points x of measure not greater than K/M such that $s_{(c,x)}(x) = s(x)$ is of constant sign on T and that

$$\Big| \int_T s(x)\,dx \Big| > \frac{1}{M} \sup_{f \in L_0} r\big(f, (c_0, x_0)\big) = \int_{\{x: s_0(x) > a_0\}} s_0(x)\,dx.$$

Because of the equality

$$\operatorname{mes}\{x: s(x) > 0\} + \operatorname{mes}\{x: s(x) < 0\} = 1$$

at least one of the sets $\{x: s(x) > 0\}$ and $\{x: s(x) < 0\}$ has measure greater than or equal to $1/2$. Suppose we have

$$\operatorname{mes}\{x: s(x) > 0\} > \tfrac{1}{2}. \tag{2.14}$$

The plane set $\{(x, y): s(x) > 0, 0 < y < s(x)\}$, the area of which is equal to

$$\int_{\{x: s(x) > 0\}} s(x)\,dx,$$

contains certainly (see fig. 2.3) rectangular isosceles triangles

$$\{(x, y)\colon x_i < x < x_i + b_i,\ 0 < y < s(x_i) - (x - x_i)\},$$

where $b_i = \min\{x_{i+1} - x_i,\ \max(0, s(x_i))\}$ is the length of perpendicular sides of such a triangle. The joint area of these triangles is thus equal to $\frac{1}{2}(b_1^2 + \ldots + b_n^2)$. Moreover, we have the relation

$$b_1 + \ldots + b_n = \operatorname{mes}\{x\colon s(x) > 0\}.$$

By putting $b_i' = b_i/(b_1 + \ldots + b_n)$ we arrive at the relation $b_1' + \ldots + b_n' = 1/2$.

Now, analogous triangles for the function $s_0(x)$ are all congruent and the length of their perpendicular sides is $1/2n$. As x^2 is a convex function, we have

$$(2.15) \qquad (b_1'^2 + \ldots + b_n'^2) \geqslant n\left(\frac{1}{2n}\right)^2,$$

the inequality being sharp unless all b_i' are equal.

Therefore if $K/M \geqslant 1/2$, we have for $T = \bigcup_{i=1}^{n} \{x\colon x_i < x < x_i + b_i'\}$

$$\int_T s(x)\,dx > \frac{1}{2}(b_1'^2 + \ldots + b_n'^2) \geqslant n\left(\frac{1}{2n}\right)^2 = \int_{\{x: s_0(x) > \alpha_0\}} s_0(x)\,dx$$

with $\alpha_0 = 0$, which implies the assertion of lemma 7.

If $K/M < 1/2$, then $\alpha_0 > 0$ and

$$\{x\colon s_0(x) > \alpha_0\} = \bigcup_{i=1}^{n}\left\{x\colon \frac{2i-1}{2n} < x < \frac{2i-1}{2n} + \frac{K}{nM}\right\},$$

and consequently

$$\int_{\{x: s_0(x) > \alpha_0\}} s_0(x)\,dx = n\cdot\frac{1}{2}\left\{\left(\frac{1}{2n}\right)^2 - \left(\frac{1}{2n}\right)^2\left(1 - \frac{2K}{M}\right)^2\right\} = \frac{n}{2}\left(\frac{1}{2n}\right)^2(1 - \varepsilon^2)$$

with $\varepsilon = 1 - 2K/M$. If we choose

$$T = \bigcup_{i=1}^{n}\left\{x\colon x_i < x < x_i + \frac{2K}{M} b_i'\right\},$$

we will have

$$\int_T s(x)\,dx > \frac{1}{2}(1 - \varepsilon^2)(b_1'^2 + \ldots + b_n'^2) \geqslant \frac{n}{2}\left(\frac{1}{2n}\right)^2(1 - \varepsilon^2) = \int_{\{x: s_0(x) > \alpha_0\}} s_0(x)\,dx,$$

which again implies the assertion of lemma 7.

If (2.14) does not hold, we have either

$$\operatorname{mes}\{x\colon s(x) < 0\} > 1/2,$$

or

$$\operatorname{mes}\{x\colon s(x) > 0\} = \operatorname{mes}\{x\colon s(x) < 0\} = 1/2.$$

In the first case the proof is essentially the same, the main change consisting in that we consider now rectangular isosceles triangles contained in the set $\{(x, y)\colon s(x) < 0,\ s(x) < y < 0\}$.

In the other case we base our arguments on that one of the sets $\{x\colon s(x) > 0\}$ and $\{x\colon s(x) < 0\}$ for which not all lengths of perpendicular sides of the considered triangles are equal, and we use the sharpness of inequality (2.15) for non equal b_i''s. If there is no such set among them, we obviously have $(\boldsymbol{c}, \boldsymbol{x}) = (\boldsymbol{c}_0, \boldsymbol{x}_0)$.

Lemma 7 is thus completely proved.

Theorem 2 is an immediate consequence of the four lemmas 4-7.

3. Integration with respect to a finite absolutely continuous measure. In this section we shall reword theorems of section 2 so as to get formulations pertinent to approximate integration with respect to an arbitrary finite absolutely continuous measure. This will enable us to draw conclusions about the optimal location of measurement points in cases I and II of section 1.

Let $\varphi(t)$ be a probability density on the real line $-\infty < t < \infty$. Let G be the class of non-decreasing functions $g(t)$ defined for $-\infty < t < \infty$, with values in the interval $0 \leqslant g(t) \leqslant K$ and such that for any $t' < t''$ we have $g(t') = g(t'')$ if $\int_{t'}^{t''} \varphi(t)\,dt = 0$. Let A be the set of vectors $(\boldsymbol{c}, \boldsymbol{t}) = (c_1, \ldots, c_n; t_1, \ldots, t_n)$ with $c_1 + \ldots + c_n = 1$ and $-\infty < t_1 < \ldots < t_n < \infty$. Consider the estimation game between nature and statistician, where nature chooses g from G and the statistician chooses $(\boldsymbol{c}, \boldsymbol{t})$ from A. Let

$$(3.1) \qquad \varrho(g, (\boldsymbol{c}, \boldsymbol{t})) = \left|\int_{-\infty}^{\infty} g(t)\varphi(t)\,dt - (c_1 g(t_1) + \ldots + c_n g(t_n))\right|$$

be the payoff in this game.

Consider now the transformation of the line $-\infty < t < \infty$ onto the segment $0 < x < 1$ defined by

$$(3.2) \qquad x = x(t) = \int_{-\infty}^{t} \varphi(u)\,du,$$

and the transformation of functions $g \in G$ into functions $f(x)$, $0 < x < 1$, defined by the formula

$$f(x) = g(\zeta_x), \tag{3.3}$$

where ζ_x is any root of the equation

$$\int_{-\infty}^{\zeta_x} \varphi(u)\,du = x. \tag{3.4}$$

Because of the monotonicity and boundedness of $g(t)$ the function $f(x)$ has finite limits at $x = 0$ and $x = 1$. So (3.3) defines a one-to-one transformation of the class G onto the class H of theorem 1. Moreover, if $f \in H$ corresponds to $g \in G$ by (3.3) and $(\boldsymbol{c}, \boldsymbol{x}) \in B$ corresponds to $(\boldsymbol{c}, \boldsymbol{t}) \in A$ by (3.2), we have

$$\int_{-\infty}^{\infty} g(t)\varphi(t)\,dt = \int_0^1 f(x)\,dx, \qquad c_1 g(t_1) + \ldots + c_n g(t_n) = c_1 f(x_1) + \ldots + c_n f(x_n), \tag{3.5}$$

and, consequently,

$$\varrho\big(g, (\boldsymbol{c}, \boldsymbol{t})\big) = r\big(f, (\boldsymbol{c}, \boldsymbol{x})\big). \tag{3.6}$$

This enables us to reword theorem 1 in the following way:

THEOREM 1'. *In the estimation game between nature, which chooses g from G, and the statistician, who chooses $(\boldsymbol{c}, \boldsymbol{t})$ from A, with the payoff $\varrho(g, (\boldsymbol{c}, \boldsymbol{t}))$ defined by* (3.1), *a minimax strategy of the statistician is a vector $(\boldsymbol{c}_0, \boldsymbol{t}_0)$ defined by*

$$\boldsymbol{c}_0 = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right), \qquad \boldsymbol{t}_0 = (\zeta_{1/2n}, \ldots, \zeta_{(2n-1)/2n}), \tag{3.7}$$

and the minimax risk of the statistician is given by

$$\inf_{(\boldsymbol{c},\boldsymbol{t})\in A} \sup_{g\in G} \varrho\big(g, (\boldsymbol{c}, \boldsymbol{t})\big) = K/2n. \tag{3.8}$$

Similarly, given the probability density $\varphi(t)$, $-\infty < t < \infty$, we can consider the class A of non-decreasing functions $g(t)$, $-\infty < t < \infty$, with values in the interval $0 \leqslant g(t) \leqslant K$ which fulfil the Lipschitz condition relative to the probability distribution determined by $\varphi(t)$, i. e. such that for any $t' < t''$ we have the inequality

$$g(t'') - g(t') \leqslant M \int_{t'}^{t''} \varphi(t)\,dt. \tag{3.9}$$

We see that the class A of functions $g(t)$, $-\infty < t < \infty$, is transformed by (3.3) onto the class L of functions $f(x)$, $0 \leqslant x \leqslant 1$, of theorem 2, and the set A is transformed by (3.2) onto B of theorem 2 in such a manner that we have (3.6) if f is the image of g and x is the image of t. Thus we can reword theorem 2 in the following way:

THEOREM 2'. *In the estimation game, where nature chooses g from A and the statistician chooses $(\boldsymbol{c}, \boldsymbol{t})$ from A, while the payoff $\varrho(g, (\boldsymbol{c}, \boldsymbol{t}))$ is given by* (3.1), *a minimax strategy of the statistician is given by $(\boldsymbol{c}_0, \boldsymbol{t}_0)$ as defined by* (3.7), *and the minimax risk of the statistician is*

$$\inf_{(\boldsymbol{c},\boldsymbol{t})\in A} \sup_{g\in A} \varrho\big(g, (\boldsymbol{c}, \boldsymbol{t})\big) = \begin{cases} \dfrac{1}{2n}\dfrac{M}{4} & \text{if} \quad M \leqslant 2K, \\[2ex] \dfrac{K}{2n}\left(1 - \dfrac{K}{M}\right) & \text{if} \quad 2K \geqslant M. \end{cases} \tag{3.10}$$

COROLLARY 1. *If by conditions of theorem 1' or 2' we have*

$$\varphi(t) = \begin{cases} 3t^2 & \text{for} \quad 0 < t < 1, \\ 0 & \text{otherwise} \end{cases}$$

(see case I of section 1), *then the best system of nodes $t_1, \ldots, t_n$ is given by*

$$t_i = \sqrt[3]{(2i-1)/2n}, \qquad i = 1, 2, \ldots, n.$$

COROLLARY 2. *If by conditions of theorem 1' or 2' we have*

$$\varphi(t) = \begin{cases} 2t & \text{for} \quad 0 < t < 1, \\ 0 & \text{otherwise} \end{cases}$$

(see case II of section 1), *then the best system of nodes $t_1, \ldots, t_n$ is given by*

$$t_i = \sqrt{(2i-1)/2n}, \qquad i = 1, 2, \ldots, n.$$

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P R O B L È M E S

P 101, R 4. Résultats ultérieurs ont été obtenus par Znám [1].

II. 3-4, p. 301; R 1, V. 1, p. 116; R 2, VI, p. 329; R 3, VII. 2, p. 307 et 308.

[1] S. Znám, *On a combinatorical problem of K. Zarankiewicz*, Colloquium Mathematicum 11 (1963), p. 81-84

P 356, R 2. Le même résultat que celui de Lax, cité dans **P 356, R 1** [2], a été etabli avant lui par Trzeciakiewicz [3].

IX. 1, p. 165.

[2] **P 356, R 1,** Colloquium Mathematicum 10 (1963), p. 184.

[3] L. Trzeciakiewicz, *Remarque sur les translations des ensembles linéaires*, Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Cl. III, 25 (1932), p. 63-65.

P 361, R 2. La réponse négative signalée dans le fascicule précédent de ce volume, p. 184, est publiée dans ce fascicule [4].

IX. 1, p. 166.

[4] B. Gleichgewicht, *A remark on absolute-valued algebras*, Colloquium Mathematicum 11 (1963), p. 29-30.

P 417, R 1. Voici une solution négative, trouvée par H. Davenport et signalée par lui à l'auteur du problème:

Si $f(x, y) = (x^2+xy-y^2+1)(x^2+xy-y^2-1)$ et X est l'ensemble des nombres de Fibonacci $u_1, u_2, \ldots$, il existe pour tout $x \in X$ un $y \in X$ tel que $f(x, y) = 0$ (à savoir, $y = u_{n+1}$ lorsque $x = u_n$).

Le problème reste ouvert même lorsque W est irréductible dans le corps des nombres rationnels.

X. 1, p. 187.