On the other hand, by Lemma 4,

$$f(a_1, d_1, f(a_1, d_1, d_2)) \neq a_1.$$

Thus, $f(a_1, d_1, d_2) \neq f(d_1, a_1, d_2)$, which, according to (25) and (26), implies the equation $f(d_1, a_1, d_2) = c_2$. Since the operation f is alternating, the last equation and (27) imply

$$f(c_1, a_1, d_1) = f(a_1, d_1, c_2) = f(a_1, d_1, f(d_1, a_1, d_2)) = a_1,$$

which contradicts (24).

Now consider the case $f(a_1, d_1, d_2) = c_2$. Since the operation f is alternating, we have, by (22), the equation

$$f(c_2, a_1, d_1) = f(a_1, d_1, c_2) = f(a_1, d_1, f(a_1, d_1, d_2)) = f(a_1, d_1, d_2) = c_2,$$

which also contradicts (24). This completes the proof of the independence of each three-element subset of $[a_1, a_2, a_3]$ which does not contain the element c_2. Since a_1, a_2, a_3 are independent in the algebra $(A; \boldsymbol{F})$, to prove the Theorem it is sufficient, by Lemma 3, to show that all three-element subsets of $[a_1, a_2, a_3]$ are independent in the algebra $[a_1, a_2, a_3]$. If $c_2 \notin [a_1, a_2, a_3]$, then it is obvious. Further, if $c_2 \in [a_1, a_2, a_3]$, then every three-element subset of $[a_1, a_2, a_3]$ which does not contain c_2 is independent. Moreover, the set $\{c_1, c_2, a_1\}$ containing c_2 is independent and, by (17), is contained in $[a_1, a_2, a_3]$. Thus, by the first part of the proof ($n = 3$, $m = 1$), every three-element subset of $[a_1, a_2, a_3]$ is independent, which completes the proof of Theorem 2.

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ON FREE PRODUCTS OF $\mathfrak{m}$-DISTRIBUTIVE BOOLEAN ALGEBRAS

BY

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1. Introduction. Let $\mathfrak{m}$ be an arbitrary cardinal number and $\{\mathfrak{A}_t\}_{t\in T}$ be an indexed set of non-degenerate $\mathfrak{m}$-complete Boolean algebras. An $\mathfrak{m}$-complete Boolean algebra $\mathfrak{B}$ is said to be a *minimal* $\mathfrak{m}$*-product* of $\{\mathfrak{A}_t\}_{t\in T}$ if there exist $\mathfrak{m}$-isomorphisms

$$i_t : \mathfrak{A} \to \mathfrak{B} \quad (t \in T)$$

such that

(a) the union of all the subalgebras $i_t(\mathfrak{A}_t)$ $\mathfrak{m}$-generates $\mathfrak{B}$,

(b) the subalgebras $i_t(\mathfrak{A}_t)$, $t \in T$, are $\mathfrak{m}$-independent in $\mathfrak{B}$,

(c) the set of all meets of the form

$$\bigcap_{t\in T'} i_t(A_t) \quad \text{where} \quad A_t \in \mathfrak{A}_t, \quad T' \subset T, \ \overline{\overline{T'}} \leqslant \mathfrak{m}$$

is dense in $\mathfrak{B}$.

Christensen and Pierce [1] proved the existence of the minimal $\mathfrak{m}$-product of any indexed set of non-degenerate $\mathfrak{m}$-complete Boolean algebras (for $\mathfrak{m} = \aleph_0$ see also Sikorski [5]). They proved also that

1.1. *The minimal $\mathfrak{m}$-product of $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebras is a free $\mathfrak{m}$-distributive product of these algebras.*

We recall that an $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{B}$ is said to be a *free* $\mathfrak{m}$*-distributive product* of an indexed set $\{\mathfrak{A}_t\}_{t\in T}$ of $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebras if there exist isomorphisms

$$i_t : \mathfrak{A}_t \to \mathfrak{B} \quad (t \in T)$$

such that

(α) the union of all the subalgebras $i_t(\mathfrak{A}_t)$ $\mathfrak{m}$-generates $\mathfrak{B}$,

(β) if, for every $t \in T$, h_t is a homomorphism of $i_t(\mathfrak{A}_t)$ into any $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{C}$, then there is a homomorphism h of $\mathfrak{B}$ into $\mathfrak{C}$ which is a common extension of all the homomorphisms h_t, i. e. $h_t(A) = h(A)$ for $A \in i_t(\mathfrak{A}_t)$ (cf. Sikorski [3], p. 214).

The purpose of the present paper is to give another proof of theorem 1.1 using the following well known statements:

(A) *An* $\mathfrak{m}$*-complete Boolean algebra* $\mathfrak{B}$ *is* $\mathfrak{m}$*-distributive if and only if every* $\mathfrak{m}$*-complete subalgebra generated by at most* $\mathfrak{m}$ *elements of* $\mathfrak{B}$ *is atomic* (*i. e. is isomorphic with an* $\mathfrak{m}$*-field of sets*) (see [4], p. 82);

(B) *If* $\mathfrak{B}$ *is an* $\mathfrak{m}$*-complete Boolean algebra with a set* $\mathfrak{S}$ *of generators, then a mapping* f *of* $\mathfrak{S}$ *into any* $\mathfrak{m}$*-complete field* $\mathfrak{B}'$ *of subsets of a set* X *can be extended to an* $\mathfrak{m}$*-homomorphism of* $\mathfrak{B}$ *into* $\mathfrak{B}'$ *if and only if*

$$\bigcap_{t\in T}\varepsilon(t)A_t = \wedge \text{ implies } \bigcap_{t\in T}\varepsilon(t)f(A_t) = \wedge,$$

where $\overline{\overline{T}} \leqslant \mathfrak{m}$ *and* $\varepsilon(t) = 1$ *or* -1 *for every* $t \in T$ (see [4], p. 115).

Here $1\cdot A = A$ and $-1\cdot A = -1 =$ the complement of A.

We adopt in the present paper the terminology of [4]. In particular $\wedge$ and $\vee$ denote the zero and the unit element of a Boolean algebra respectively.

2. Minimal $\mathfrak{m}$**-products.** Let $\{\mathfrak{A}_t\}_{t\in T}$ be a fixed indexed set of non-degenerate $\mathfrak{m}$-complete Boolean algebras, let $\mathfrak{B}$ be the minimal $\mathfrak{m}$-product of $\{\mathfrak{A}_t\}_{t\in T}$, and let i_t be $\mathfrak{m}$-isomorphisms of $\mathfrak{A}_t$ into $\mathfrak{B}$, $t\in T$, such that (a), (b), (c) hold.

LEMMA 2.1. *Suppose* $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$. *For every* $t \in T$ *let* S_t *be a non-empty set and let* S *be the set of all mappings* f *of* T' *into* $\bigcup_{t\in T'} S_t$ *such that* $f(t)\in S_t$ *for every* $t \in T'$.

If $\wedge \neq A_{t,s} \in i_t(\mathfrak{A}_t)$ *for* $s \in S_t$ *and* $t\in T$, *and if*

$$(*)\qquad \bigcup_{s\in S_t}^{i_t(\mathfrak{A}_t)} A_{t,s} = \vee \quad \textit{for every} \quad t\in T,$$

then

$$(**)\qquad \bigcup_{f\in S}^{\mathfrak{B}} \bigcap_{t\in T'}^{\mathfrak{B}} A_{t,f(t)} = \vee.$$

Proof. Suppose the lemma is not true. Then there exists, by (c), a subset $T'' \subset T$, $\overline{\overline{T''}} \leqslant \mathfrak{m}$, and non-zero elements $A_t \in i_t(\mathfrak{A}_t)$, $t\in T''$, such that

$$\bigcap_{t\in T''} A_t \cap \bigcap_{t\in T'} A_{t,f(t)} = \wedge \quad \text{for every} \quad f\in S.$$

Then, by (b), $T''\cap T' \neq 0$. But, by (*), there is a mapping $f_0 \in S$ such that

$$A_t \cap A_{t,f_0(t)} \neq \wedge \quad \text{for} \quad t\in T''\cap T'.$$

This leads, however, to a contradiction because

$$\bigcap_{t\in T''} A_t \cap \bigcap_{t\in T'} A_{t,f_0(t)} = \bigcap_{t\in T''\cap T'} A_t \cap A_{t,f_0(t)} \cap \bigcap_{t\in T''-T'} A_t \cap \bigcap_{t\in T'-T''} A_{t,f_0(t)} \neq \wedge$$

by (b), $T''\cup T'$ being of a power $\leqslant \mathfrak{m}$.

LEMMA 2.2. *Let* $T' \subset T$, $\overline{\overline{T'}} \leqslant \mathfrak{m}$. *If* $\mathfrak{B}_t$ *is an* $\mathfrak{m}$*-complete atomic subalgebra of* $i_t(\mathfrak{A}_t)$ *for every* $t \in T'$, *then the* $\mathfrak{m}$*-subalgebra* $\mathfrak{B}_0$ $\mathfrak{m}$*-generated by the union of all the* $\mathfrak{B}_t$ *is atomic and it is the minimal* $\mathfrak{m}$*-product of* $\{\mathfrak{B}_t\}_{t\in T'}$.

Proof. We shall apply lemma 2.1. For this purpose let $\{A_{t,s}\}_{s\in S_t}$ be the set of all atoms of $\mathfrak{B}_t$ ($A_{t,s} \neq A_{t,s'}$ for $s \neq s'$) and let

$$\mathfrak{H} = \{\bigcap_{t\in T'} A_{t,f(t)} : f\in S\}.$$

Then

$$\bigcap_{t\in T'} A_{t,f(t)} \cap \bigcap_{t\in T'} A_{t,f'(t)} = \wedge \quad \text{for} \quad f\neq f'.$$

Therefore, by (**), the class of all the elements $A\in\mathfrak{B}$ such that both A and $-A$ are joins (in $\mathfrak{B}$) of some elements of $\mathfrak{H}$ is an $\mathfrak{m}$-subalgebra, say $\mathfrak{B}_1$, of $\mathfrak{B}$. Multiplying both sides of (**) by any element $A \in \mathfrak{B}_t$, we infer that $\mathfrak{B}_1$ contains all the subalgebras $\mathfrak{B}_t$, $t\in T$. Thus it contains $\mathfrak{B}_0$. This proves that every non-zero element in $\mathfrak{B}_0$ contains at least one element of $\mathfrak{H}$ as its subelement. Thus $\mathfrak{B}_0$ is atomic and $\mathfrak{H}$ is the set of atoms of $\mathfrak{B}_0$. Thus the first part of the lemma is proved. The second part follows from the first since the set $\mathfrak{H}$ is dense in $\mathfrak{B}_0$.

THEOREM 2.3. *The minimal* $\mathfrak{m}$*-product* $\mathfrak{B}$ *of an indexed set* $\{\mathfrak{A}_t\}_{t\in T}$ *of non-degenerate* $\mathfrak{m}$*-complete* $\mathfrak{m}$*-distributive Boolean algebras is* $\mathfrak{m}$*-distributive.*

Proof. Let $\mathfrak{S}$ be a subset of $\mathfrak{B}$ of a power $\leqslant \mathfrak{m}$ and let $\mathfrak{B}_1$ be the $\mathfrak{m}$-subalgebra of $\mathfrak{B}$ $\mathfrak{m}$-generated by $\mathfrak{S}$. Then for every $A\in\mathfrak{S}$ there exists a set $T_A \subset T$ and for every $t\in T_A$ a set $\mathfrak{S}_{t,A} \subset i_t(\mathfrak{A}_t)$, $\overline{\overline{T_A}} \leqslant \mathfrak{m}$, $\overline{\overline{\mathfrak{S}_{t,A}}} \leqslant \mathfrak{m}$, such that A is in the $\mathfrak{m}$-subalgebra $\mathfrak{m}$-generated by the union of all the sets $\mathfrak{S}_{t,A}$, $t\in T_A$. Consequently $\mathfrak{B}_1$ is an $\mathfrak{m}$-subalgebra of the $\mathfrak{m}$-subalgebra $\mathfrak{B}_0$ $\mathfrak{m}$-generated by the union of all the sets $\mathfrak{S}_{t,A}$ ($A\in\mathfrak{S}$, $t\in T_A$).

Since the Boolean algebra $i_t(\mathfrak{A}_t)$ is $\mathfrak{m}$-distributive, it follows, by (A), that the $\mathfrak{m}$-subalgebra of $i_t(\mathfrak{A}_t)$ generated by $\mathfrak{S}_{t,A}$ is atomic. Thus, by 2.2, $\mathfrak{B}_0$ is atomic, too. Consequently $\mathfrak{B}_1$ is isomorphic with an $\mathfrak{m}$-complete field of sets. This completes the proof, by (A).

3. Extensions of homomorphisms. Let $\mathfrak{A}$ be an $\mathfrak{m}$-complete Boolean algebra. For any subset $\mathfrak{R} \subset \mathfrak{A}$ let us denote by $\mathfrak{R}_\mathfrak{m}$ the $\mathfrak{m}$-subalgebra of $\mathfrak{A}$ $\mathfrak{m}$-generated by $\mathfrak{R}$.

LEMMA 3.1. *Let* f *be a mapping of an infinite set* $\mathfrak{R}_0 \subset \mathfrak{A}$ *in an* $\mathfrak{m}$*-complete Boolean algebra* $\mathfrak{B}$. *If, for every subset* $\mathfrak{R} \subset \mathfrak{R}_0$ *of a power* $\leqslant \mathfrak{m}$, *there exists an* $\mathfrak{m}$*-homomorphism* $h_\mathfrak{R}$ *of* $\mathfrak{R}_\mathfrak{m}$ *into* $\mathfrak{B}$ *such that* $h_\mathfrak{R}(A) = f(A)$ *for every* $A\in\mathfrak{R}$, *then there exists an* $\mathfrak{m}$*-homomorphism* h *of* $\mathfrak{R}_{0,\mathfrak{m}}$ *into* $\mathfrak{B}$ *which is an extension of* f.

The above lemma has been proved for $\mathfrak{m} = \aleph_0$ in [2]. The generalization for $\mathfrak{m} > \aleph_0$ is trivial.

THEOREM 3.2. *A mapping f of a set $\mathfrak{K} \subset \mathfrak{A}$ in an $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{B}$ can be extended to an $\mathfrak{m}$-homomorphism h of $\mathfrak{K}_{\mathfrak{m}}$ in $\mathfrak{B}$ if and only if for every set $\{A_t : t \in T, \overline{\overline{T}} \leqslant \mathfrak{m}\} \subset \mathfrak{K}$*

(i) $\bigcap_{t\in T} \varepsilon(t) A_t = \wedge$ *implies* $\bigcap_{t\in T} \varepsilon(t) f(A_t) = \wedge$,

where $\varepsilon(t) = 1$ or -1 for every $t \in T$.

Proof. The necessity is obvious. By lemma 3.1 we must prove the sufficiency of (i) only in the case where the power of $\mathfrak{K}$ is $\leqslant \mathfrak{m}$.

In this case, however, the power of $f(\mathfrak{K})$ is also $\leqslant \mathfrak{m}$. Hence $f(\mathfrak{K})_{\mathfrak{m}}$ is isomorphic with an $\mathfrak{m}$-complete field of sets, by $\mathfrak{m}$-distributivity of $\mathfrak{B}$.

Therefore, by (B), the mapping f of $\mathfrak{K}$ into $f(\mathfrak{K})_{\mathfrak{m}}$ can be extended to an $\mathfrak{m}$-homomorphism

$$h_{\mathfrak{K}} : \mathfrak{K}_{\mathfrak{m}} \to f(\mathfrak{K})_{\mathfrak{m}},$$

i. e. to an $\mathfrak{m}$-homomorphism $h_{\mathfrak{K}} : \mathfrak{K}_{\mathfrak{m}} \to \mathfrak{B}$, q. e. d.

4. The proof of theorem 1.1. Let $\{\mathfrak{A}_t\}_{t\in T}$ be an indexed set of non-degenerate $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebras. Let $\mathfrak{B}$ be the minimal $\mathfrak{m}$-product of these algebras. By 2.3, $\mathfrak{B}$ is $\mathfrak{m}$-distributive.

Let $\mathfrak{C}$ be any $\mathfrak{m}$-complete $\mathfrak{m}$-distributive Boolean algebra. By the definition of free $\mathfrak{m}$-distributive product of an indexed set of Boolean algebras (see the introduction) it remains to prove that if, for every $t \in T$, h_t is an $\mathfrak{m}$-homomorphism of $i_t(\mathfrak{A}_t)$ into $\mathfrak{C}$, then there exists an $\mathfrak{m}$-homomorphism h of $\mathfrak{B}$ into $\mathfrak{C}$ which is a common extension of all the homomorphisms h_t.

This follows, however, immediately from 3.2. Condition (i) is satisfied since the subalgebras $i_t(\mathfrak{A}_t)$ of $\mathfrak{B}$ are $\mathfrak{m}$-independent.

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MINIMAL EXTENSIONS OF WEAKLY DISTRIBUTIVE BOOLEAN ALGEBRAS

BY

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Introduction. Pierce [2] has proved two important theorems on minimal extensions of $\mathfrak{m}$-distributive Boolean algebras. The purpose of the present paper is to generalize those theorems to weakly $\mathfrak{m}$-distributive Boolean algebras.

Terminology and notation. The symbol $\bigcup$ will be used both for the Boolean join and for the set-theoretical union. The symbol $\bigcap$, similarly, will be used both for the Boolean meet and for the set-theoretical intersection. The zero element of a Boolean algebra will be denoted by 0 and the unit element by 1.

A Boolean algebra and the set of all its elements will be denoted by the same letter.

A subset A of a Boolean algebra B is said to be a *covering* of B if $\bigcup_{a\in A} a = 1$.

A covering A of a Boolean algebra B is said to be $\mathfrak{m}$-*covering* of B if $\overline{\overline{A}} \leqslant \mathfrak{m}$, where $\overline{\overline{A}}$ denotes the cardinal number of A. A covering or $\mathfrak{m}$-covering A is called *partition*, respectively $\mathfrak{m}$-*partition* if elements of A are disjoint.

If A and C are subsets of a Boolean algebra B, we say that A *refines* C, if for every $a \in A$ there exists $c \in C$ such that $a \subset c$; we say that A *weakly refines* C if for every $a \in A$ there exists a finite sequence

$$(c_1, c_2, \ldots, c_k) \subset C$$

such that $a \subset \bigcup_{i=1}^{k} c_i$.

A subalgebra B_2 of a Boolean algebra B_1 is said to be an $\mathfrak{m}$-*regular subalgebra* of B_1, when for every set $A \subset B_2$, $\overline{\overline{A}} \leqslant \mathfrak{m}$, if the join $\bigcup_{a\in A} a$ exists in B_2, it is also the join of this set in B_1. If B_2 is an $\mathfrak{m}$-regular subalgebra