COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

TAUBERIAN THEOREMS FOR CESÀRO SUMS

BY

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1. INTRODUCTION AND NOTATION

As in Varadarajan's paper [7] to which this is a sequel, $\{S_n^a\}$ is the sequence of Cesàro sums of order a, of a real sequence $\{s_n\}$, but now we work with $a > -1$ and not necessarily an integer. Theorem II of this paper, with $p+1$ changed to p, is a Tauberian theorem for the Cesàro sums $S_n^{p+\delta}$, where $p = 1, 2, \ldots$, $0 < \delta \leqslant 1$, which was considered by Varadarajan in the case $\delta = 1$ ([7], Theorem B). Corollary II under Theorem II, similarly changed and holding for $p = 1, 2, 3, \ldots$, has a complement in Theorem I for the case $p = 0$. Theorem I itself includes several special cases stated as corollaries. Corollaries I_1 and I_2 are familiar results due to Hardy and Littlewood. Corollary I_3 in its alternative (b) is essentially a theorem given by Dixon and Ferrar ([3], Theorem I). Corollary I_4 is a result given by Lord ([4], Lemma 11) and applied by him to prove a Tauberian theorem for passage from Borel summability to Cesàro summability of a certain order. Corollary I_4 can also be derived from a result suggested by Kuttner to Rajagopal ([5], Lemma 5). But Kuttner's result, while being broadly similar to Theorem I, is not as simple and readily applicable as Theorem I.

We first recall that S_n^a $(n = 0, 1, \ldots)$ for all $a > -1$ is given by the formula

$$S_n^a = \sum_{\nu=0}^{n} A_{n-\nu}^{a-1} s_\nu, \tag{1}$$

where A_ν^{a-1} = coefficient of x^ν in $(1-x)^{-a}$ $(|x| < 1)$.

If we define $S_n^{-1} = s_n - s_{n-1}$, then, for a, β such that $a \geqslant -1$, $\beta \geqslant -1$, $a+\beta \geqslant -1$, the sequence of Cesàro sums of order $a+\beta$ of $\{s_n\}$ is the same as the sequence of Cesàro sums of order β of $\{S_n^a\}$. Furthermore, summability (C, a), $a > -1$, of $\{s_n\}$ to sum l is the relation $\Gamma_{(a+1)} S_n^a / n^a \to l$ $(n \to \infty)$.

We require the following finite differences of $\{s_n\}$ in which h, k are positive integers, $p=1,2,\ldots,\ 0<\delta\leqslant 1$:

$$\Delta_h^0 s_n = s_n,\qquad \Delta_h^1 s_n = s_{n+h}-s_n,\qquad \Delta_h^p s_n = \Delta_h^1\Delta_h^{p-1}s_n;$$

$$\Delta_{-k}^0 s_n = s_n,\qquad \Delta_{-k}^1 s_n = s_n - s_{n-k},\qquad \Delta_{-k}^p s_n = \Delta_{-k}^1\Delta_{-k}^{p-1}s_n;$$

$$(2)\quad\begin{cases}\text{(i)} & \Delta_h^{p+\delta}s_n = \delta\displaystyle\sum_{\nu=1}^{h}\frac{\Gamma(h-\nu+\delta)}{\Gamma(h-\nu+1)}\,\Delta_h^p S_{n+\nu}^{-\delta};\\[2ex] \text{(ii)} & \Delta_{-k}^{p+\delta}s_n = \delta\displaystyle\sum_{\nu=1}^{k}\frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)}\,\Delta_{-k}^p S_{n+1-\nu}^{-\delta}\qquad (n+1\geqslant(p+1)k).\end{cases}$$

The differences of fractional order $p+\delta$ given above were introduced by Dr. Bosanquet ([2], § 3.1) who has since pointed out, in a letter to Prof. Rajagopal, a misprint in (2), (ii), as originally given by him. The misprint, now corrected for the first time in (2), (ii), consists in the appearance originally of $\Gamma(k-\nu+\delta)$ and $\Gamma(k-\nu+1)$ respectively where $\Gamma(\nu-1+\delta)$ and $\Gamma(\nu)$ now occur. This misprint is corrected also in the statement of Bosanquet's result given below as Lemma C, (ii).

2. LEMMAS

LEMMA A ([1], Theorem 1; or [2], Lemma 5). *If* $0<m\leqslant n$ *and* $0<\delta\leqslant 1$, *then*

$$\Big|\sum_{\nu=0}^{m}A_{n-\nu}^{\delta-1}s_\nu\Big|\leqslant\max_{0\leqslant\mu\leqslant m}|S_\mu^\delta|.$$

LEMMA B (Cf. [1], Theorems 2, 3). *If* $d_1, d_2, \ldots$ *is a positive monotonic sequence,* p *is a non-negative integer,* $0<m\leqslant m'\leqslant n$ *and* $0<\delta\leqslant 1$, *then*

$$\Big|\sum_{\nu=m}^{m'}A_\nu^{\delta-1}d_\nu S_{n-\nu}^p\Big|\leqslant 2\max_{m\leqslant\nu\leqslant m'}d_\nu\cdot\max_{0\leqslant\mu\leqslant n}|S_\mu^{p+\delta}|.$$

Proof. The proof which follows is for monotonic increasing $\{d_n\}$. The proof for monotonic decreasing $\{d_n\}$ is similar and omitted. We have

$$\Big|\sum_{\nu=m}^{m'}A_\nu^{\delta-1}d_\nu S_{n-\nu}^p\Big| = \Big|A_{m'}^{\delta-1}d_{m'}S_{n-m'}^p + A_{m'-1}^{\delta-1}d_{m'-1}S_{n-m'+1}^p+\ldots+A_m^{\delta-1}d_m S_{n-m}^p\Big|$$

$$\leqslant d_{m'}\max_{1\leqslant k\leqslant m'-m+1}\big|A_{m'}^{\delta-1}S_{n-m'}^p + A_{m'-1}^{\delta-1}S_{n-m'+1}^p+\ldots\text{ to }k\text{ terms}\big|$$

$$\leqslant d_{m'}\max_{1\leqslant k\leqslant m'-m+1}\Big\{\Big|\sum_{\nu=0}^{n-m'+k}A_{n-\nu}^{\delta-1}S_\nu^p\Big| + \Big|\sum_{\nu=0}^{n-m'}A_{n-\nu}^{\delta-1}S_\nu^p\Big|\Big\},$$

where the second term on the right side is absent if $m'=n$. The required result now follows from Lemma A.

LEMMA C (Cf. [2], Theorem 5). *For positive integers* h, k, $p=1,2,\ldots,$ $0<\delta\leqslant 1$,

$$\text{(i)}\qquad \Delta_h^{p+\delta}S_n^{p+\delta} = \delta\sum_{\nu_0=1}^{h}\frac{\Gamma(h-\nu_0+\delta)}{\Gamma(h-\nu_0+1)}\sum_{\nu_1=1}^{k}\cdots\sum_{\nu_p=1}^{h}s_{n+\nu_0+\ldots+\nu_p},$$

$$\text{(ii)}\qquad \Delta_{-k}^{p+\delta}S_n^{p+\delta} = \delta\sum_{\nu_0=1}^{k}\frac{\Gamma(\nu_0-1+\delta)}{\Gamma(\nu_0)}\sum_{\nu_1=1}^{k}\cdots\sum_{\nu_p=1}^{k}s_{n+p+1-\nu_0-\nu_1-\ldots-\nu_p}$$

if $n>(p+1)k$.

3. THEOREMS

THEOREM I. *Let* $W(x)$, $V(x)$ *be positive functions of* $x>0$, *such that*

$$(3)\quad\begin{cases}\text{(i)} & W(x)\ \textit{is monotonic increasing and unbounded},\\ \text{(ii)} & V(x')/V(x)<H\ \textit{if}\ 0<|x'-x|<\eta x\ (\eta<1),\\ \text{(iii)} & \{W(x)/V(x)\}^{1/\delta}=O(x)\ \textit{as}\ x\to\infty\ \textit{where}\ 0<\delta\leqslant 1.\end{cases}$$

Then

$$(4)\qquad S_n^\delta = o\{W(n)\}\quad\textit{as}\quad n\to\infty,$$

$$(5)\qquad s_n = O\{V(n)\}\quad\textit{as}\quad n\to\infty,$$

together imply, for any r *such that* $0<r<\delta$,

$$(6)\qquad S_n^r = o[\{V(n)\}^{1-r/\delta}\{W(n)\}^{r/\delta}]\quad\textit{as}\quad n\to\infty.$$

Proof. By definition (1),

$$(7)\qquad \Gamma(r)S_n^r = \Big(\sum_{\nu=0}^{n-h}+\sum_{\nu=n-h+1}^{n}\Big)\frac{\Gamma(n-\nu+r)}{\Gamma(n-\nu+1)}s_\nu$$

$$= \sum_{\nu=0}^{n-h}\frac{\Gamma(n-\nu+\delta)}{\Gamma(n-\nu+1)}d_{n-\nu}s_\nu + \sum_{\nu=n-h+1}^{n}\frac{\Gamma(n-\nu+r)}{\Gamma(n-\nu+1)}s_\nu$$

$$= T_1+T_2\ \text{(say)},\qquad\text{where}\qquad d_\nu = \frac{\Gamma(\nu+r)}{\Gamma(\nu+\delta)}.$$

By Lemma B, (4), and (3) (i),

$$(8)\qquad |T_1| = \Gamma(\delta)\Big|\sum_{\nu=h}^{n} A_\nu^{\delta-1} d_\nu s_{n-\nu}\Big| \leqslant 2\Gamma(\delta) d_h \max_{0\leqslant\mu\leqslant n} |S_\mu^\delta|$$
$$< 2\Gamma(\delta)\frac{\Gamma(h+r)}{\Gamma(h+\delta)}\varepsilon^\delta W(n)$$

for all sufficiently large n. Also, by (5) and (3), (ii),

$$(9)\qquad |T_2| \leqslant K \max_{n-h+1\leqslant\nu\leqslant n} V(\nu) \sum_{\nu=n-h+1}^{n}\frac{\Gamma(n-\nu+r)}{\Gamma(n-\nu+1)} < KHV(n)\frac{\Gamma(h+r)}{r\Gamma(h)},$$

provided that $h < \eta n$. Now use (8) and (9) in (7), observing that, for any choice of h which tends to ∞,

$$\frac{\Gamma(h+r)}{\Gamma(h+\delta)} \sim \frac{1}{h^{\delta-r}},\qquad \frac{\Gamma(h+r)}{\Gamma(h)} \sim \frac{1}{h^r}$$

in (8) and (9). Next choose $h =$ integral part of $\varepsilon\{W(n)/V(n)\}^{1/\delta}$ with ε so small that $h < \eta n$, this being possible by (3)(iii). Then (7) gives for all sufficiently large n (and h),

$$S_n^r < K'\left\{\frac{\varepsilon^\delta}{h^{\delta-r}} W(n) + h^r V(n)\right\} \sim 2K'\varepsilon^r\{V(n)\}^{1-r/\delta}\{W(n)\}^{r/\delta},$$

which is the required result (6) since ε can be chosen arbitrarily small.

The case $\delta = 1$, $W(x) \equiv x$, $V(x) \equiv 1$ of Theorem I is the following well-known result:

COROLLARY I_1. *A sequence s_n bounded and summable $(C, 1)$ to 0 is summable (C, r) to 0 for every $r > 0$.*

Theorem I for the sequence $s_n^* \equiv s_n - s_{n-1} = S_n^{-1}$ instead of the sequence s_n, with $\delta = 1$, $W(x) \equiv 1$, $V(x) \equiv 1/x$, gives another well-known result:

COROLLARY I_2. *A sequence s_n, convergent to 0 and such that $s_n - s_{n-1} = O(1/n)$ is summable $(C, -1+r)$ to 0 for every r such that $0 < r < 1$.*

COROLLARY I_3. *Suppose that, in Theorem* I, *we assume* (3) (i) *with or without the unboundedness of $W(n)$,* (3) (iii), *along with either* (a) (3) (ii) *or* (b) *the restriction that $V(x)$ is monotonic increasing. Then Theorem* I *can be restated with o changed to O in* (4), *O changed to O_L in* (5) *and o changed to O_L in* (6).

(Here we say (see e.g. [4]) that $f = O_L(\varphi)$, φ being positive, if a positive constant K exists such that $f > -K\varphi$, and, similarly, that $f = O_R(\varphi)$, if $f < K\varphi$).

COROLLARY I_4. *If $a \geqslant -1$, $p \geqslant \frac{1}{2}$,*

$$S_n^{a+1} = O(n^{a+p+1/2}),\qquad S_n^a = O_L(n^{a+p}),\qquad as\quad n\to\infty,$$

then, for any r such that $0 < r < 1$,

$$S_n^{a+r} = O_L(n^{a+p+r/2}).$$

Lord deduces Corollary I_4 from Corollary I_3 with alternative (b), where $V(x)$ is monotonic increasing, taking in the latter $s_n^* \equiv S_n^a$ instead of s_n, $\delta = 1$, $W(x) \equiv x^{a+p+1/2}$, $V(x) \equiv x^{a+p}$. This deduction is invalid since $V(x)$ is not necessarily monotonic increasing, our assumption being only that $a+p \geqslant -\frac{1}{2}$. However, the same deduction from Corollary I_3 with alternative (a) is valid.

THEOREM II. *Let $W(x)$, $V(x)$ be positive functions of $x > 0$ such that*

$$(10)\qquad \begin{cases} \text{(i)} & W(x) \text{ is monotonic increasing and unbounded},\\ \text{(ii)} & W(x')/W(x) < H \text{ if } 0 < x'-x < \theta x\ (\theta < 1),\\ \text{(iii)} & V(x')/V(x) < H \text{ if } 0 < |x'-x| < \eta x\ (\eta < 1),\\ \text{(iv)} & \{W(x)/V(x)\}^{1/(p+\delta+1)} = O(x) \text{ as } x\to\infty,\end{cases}$$

where p is a non-negative integer, $0 < \delta \leqslant 1$. Then

$$(11)\qquad S_n^{p+\delta+1} = o\{W(n)\}\qquad as\quad n\to\infty,$$

$$(12)\qquad s_n = O_L\{V(n)\}\qquad as\quad n\to\infty,$$

together imply

$$(13)\qquad S_n^1 = o\left[\{V(n)\}^{1-\frac{1}{p+\delta+1}}\{W(n)\}^{\frac{1}{p+\delta+1}}\right]\qquad as\quad n\to\infty.$$

Proof. This has a similarity to the proofs of Theorem I and Varadarajan's special case of Theorem II. However, it requires certain additional considerations which are brought out in the following demonstration of the partial conclusion given by (13) with o_L instead of o.

In (2) (ii), let $S_n^{-\delta}$ be replaced by S_n^{p+1} and hence s_n by $S_n^{p+1+\delta}$. Then, with the definition of A_n^{a-1} in (1), we have

$$(14)\qquad \Delta_{-k}^{p+\delta} S_n^{p+1+\delta} = \delta\sum_{\nu=1}^{k}\frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)}\Delta_{-k}^{p} S_{n+1-\nu}^{p+1}\qquad (n+1 \geqslant (p+1)k)$$
$$= \delta\sum_{\nu=1}^{k}\frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)}\sum_{m=0}^{p} A_m^{-p-1} S_{n+1-\nu-mk}^{p+1}$$
$$= \delta\sum_{m=0}^{p} A_m^{-p-1}\sum_{\nu=1}^{k}\frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)} S_{N-\nu}^{p+1}\qquad (N = n+1-mk),$$

where the inner sum is

$$\sum_{\nu=1}^{k}\frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)}S_{N-\nu}^{p+1}=\Gamma(\delta)\sum_{\nu=1}^{k}A_{\nu}^{\delta-1}d_{\nu}S_{N-\nu}^{p+1},$$

$d_{\nu}=(\nu-1+\delta)/\nu$ is monotonic increasing in ν.

Hence, by Lemma B,

$$\Big|\sum_{\nu=1}^{k}\frac{\Gamma(\nu-1+\delta)}{\Gamma(\nu)}S_{N-\nu}^{p+1}\Big|\leqslant 2\Gamma(\delta)\frac{k-1+\delta}{k}\max_{0\leqslant\mu\leqslant N}|S_{\mu}^{p+1+\delta}|$$

$$\leqslant 2\Gamma(\delta)\max_{0\leqslant\mu\leqslant n+1}|S_{\mu}^{p+1+\delta}|=o\{W(n+1)\}\quad\text{as}\quad n\to\infty,$$

the last step following from (11) and (10) (i). Using this step in (14) and then using 10 (ii), we get

$$(15)\qquad |\Delta_{-k}^{p+\delta}S_{n}^{p+1+\delta}|\leqslant\delta\Big(\sum_{m=0}^{p}|A_{m}^{-p-1}|\Big)o\{W(n+1)\}=o\{W(n)\}.$$

Next, in Lemma C (ii), let s_n be replaced by S_n^1 and hence $S_n^{p+\delta}$ by $S_n^{p+\delta+1}$. Then we easily get

$$(16)\qquad k^{p}\frac{\Gamma(k+\delta)}{\Gamma(k)}S_{n}^{1}=\Delta_{-k}^{p+\delta}S_{n}^{p+1+\delta}+$$

$$+\delta\sum_{\nu_0=1}^{k}\frac{\Gamma(\nu_0-1+\delta)}{\Gamma(\nu_0)}\sum_{\nu_1=1}^{k}\ldots\sum_{\nu_p=1}^{k}(S_n^1-S^1_{n+p+1-\nu_0-\nu_1-\ldots-\nu_p})=I_1+I_2\ \text{(say)}.$$

In (16), the innermost sum of I_2 is, by (12) and (10), (iii),

$$(17)\qquad s_{n+p+2-\nu_0-\ldots-\nu_p}+\ldots+s_{n-1}+s_n$$

$$\geqslant -K\{V(n+p+2-\nu_0-\ldots-\nu_p)+\ldots+V(n)\}>-KH(p+1)kV(n)$$

for all large n, if k is chosen so that we have, for all large n,

$$(18)\qquad n-n'=n-(n+p+2-\nu_0-\ldots-\nu_p)$$

$$\leqslant(p+1)k-p-2<(p+1)k<\eta n\qquad(0<\eta<1).$$

Now we use in (16) the estimate for I_1 from (15) and the estimate for the innermost sum of I_2 from (17). (16) then gives us, for all large n,

$$(19)\qquad \frac{k^p\Gamma(k+\delta)}{\Gamma(k)}S_n^1>-\varepsilon^{p+1+\delta}W(n)-k^p\frac{\Gamma(k+\delta)}{\Gamma(k)}KH(p+1)kV(n),$$

ε being any given small positive number. We next choose k (as a function of n) so that

$$(20)\qquad k=\text{integral part of}\ \ \varepsilon\Big\{\frac{W(n)}{V(n)}\Big\}^{\frac{1}{p+1+\delta}},$$

this choice of k ensuring that $k=\varepsilon O(n)$ by (10), (iv), and hence that the choice of k in (18) is not violated. After dividing both sides of (19) by $k^p\Gamma(k+\delta)/\Gamma(k)\sim k^{p+\delta}$ $(k\to\infty)$, we use the choice of k in (20), obtaining

$$(21)\qquad S_n^1>-K'\Big[\varepsilon^{p+1+\delta}W(n)\cdot\frac{1}{\varepsilon^{p+\delta}}\Big\{\frac{V(n)}{W(n)}\Big\}^{\frac{p+\delta}{p+\delta+1}}+\varepsilon\Big\{\frac{W(n)}{V(n)}\Big\}^{\frac{1}{p+\delta+1}}V(n)\Big]$$

$$=-2K'\varepsilon\{V(n)\}^{1-\frac{1}{p+\delta+1}}\{W(n)\}^{\frac{1}{p+\delta+1}}$$

for all large n. This gives us (13) with O_L instead of O. Thus it only remains to prove (13) with o_R instead of o, using (2), (i), and Lemma C (i), instead of (2) (ii), and Lemma C (ii), respectively in the above proof.

(10) (iii), and (14) in the form which involves $W(n+1)$ show that (10) (ii), is superfluous for the preceding proof modified so as to yield (21) with $V(n+1)$ and $W(n+1)$ instead of $V(n)$ and $W(n)$. If then (12) is assumed with O instead of O_L, the preceding proof, modified as stated, is applicable to both s_n and $-s_n$, and we get the following corollary:

COROLLARY II. *If, in Theorem* II, *assumption* (10) (ii), *is omitted and assumption* (12) *is strengthened by the substitution of O for O_L, then conclusion* (13) *will become*

$$(13')\qquad S_n^1=o\Big[\{V(n+1)\}^{1-\frac{1}{p+\delta+1}}\{W(n+1)\}^{\frac{1}{p+\delta+1}}\Big].$$

Theorem II is a Cesàro version of a theorem for Riesz means proved by Rajagopal and Minakshisundaram ([6], Theorem 1). Theorem II with $s_n^*=s_n-s_{n-1}=S_n^{-1}$ instead of s_n, $p+\delta=r$, $W(x)=x^\beta$, $V(x)=x^\alpha$, is essentially a theorem of Bosanquet ([2], Theorem 6). The special case of Theorem II with $W(x)=x^{p+\delta+1}$, $V(x)=1$, enables us to change the hypothesis of summability $(C,1)$ of s_n in Corollary I_1 to the hypothesis of summability (C,r') for some $r'>0$. The special case of Theorem II, with $s_n^*=s_n-s_{n-1}$ instead of s_n, $W(x)=x^{p+\delta}$, $V(x)=1/x$, shows that we may change the hypothesis of convergence of s_n in Corollary I_2 to the hypothesis of summability (C,r') for some $r'>0$.

I wish to express my sincere thanks to Professor C. T. Rajagopal for his kind help in the preparation of this paper.

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Reçu par la Rédaction le 14. 1. 1963

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

À PROPOS D'UN THÉORÈME DE BÔCHER SUR LE WRONSKIEN

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Il y a longtemps que Bôcher a démontré (voir [1] et [2]) le théorème d'après lequel les fonctions complexes $f_1(x), \ldots, f_n(x)$ d'une variable réelle étant de classe C^{n-1} en tout point d'un intervalle ouvert I et telles que l'on a pour le wronskien l'identité

$$W(f_1, \ldots, f_{n-1}) = \begin{vmatrix} f_1 & f_2 & \cdots & f_{n-1} \\ f_1' & f_2' & \cdots & f_{n-1}' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-2)} & f_2^{(n-2)} & \cdots & f_{n-1}^{(n-2)} \end{vmatrix} \equiv 0 \quad \text{sur } I,$$

on a nécessairement aussi

$$W(f_1, \ldots, f_{n-1}, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_1' & f_2' & \cdots & f_{n-1}' & f_n' \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & f_n^{(n-1)} \end{vmatrix} \equiv 0 \quad \text{sur } I.$$

Cette communication a pour but de démontrer qu'on peut atténuer l'hypothèse de ce théorème, en admettant seulement que les fonctions $f_1, \ldots, f_n$ sont différentiables jusqu'à l'ordre $n-1$. Et voici la démonstration:

Soit x_0 un point arbitraire de l'intervalle I. Considérons deux cas suivants:

1. Les fonctions $f_1, \ldots, f_n$ sont linéairement dépendantes dans un entourage de x_0, c'est-à-dire qu'il existe un entourage E de x_0 et des nombres constants $c_1, \ldots, c_n$ tels que

$$(1) \qquad \sum_{\nu=1}^{n} c_\nu f_\nu(x) \equiv 0 \text{ sur } E \quad \text{et} \quad \sum_{\nu=1}^{n} (c_\nu)^2 > 0.$$

2. Les fonctions $f_1, \ldots, f_n$ ne sont linéairement dépendantes dans aucun entourage de x_0.