$$A_1 = \begin{pmatrix} 1/\lambda_0 & 0 & 0 & \dots & 0 \\ 0 & 1/\lambda_1 & 0 & \dots & 0 \\ 0 & 0 & 1/\lambda_2 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1/\lambda_n \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \lambda_0 a_{00} & \lambda_0 a_{01} & \dots & \lambda_0 a_{0n} \\ \lambda_1 a_{10} & \lambda_1 a_{11} & \dots & \lambda_1 a_{1n} \\ \dots & \dots & \dots & \dots \\ \lambda_n a_{n0} & \lambda_n a_{n1} & \dots & \lambda_n a_{nn} \end{pmatrix}.$$

$A_1 \epsilon H$ and A_2 are both non-exceptional matrices and $A = A_1 A_2$, q. e. d.

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Reçu par la Rédaction le 6. 2. 1963

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

ON A PROBLEM OF INTERPOLATION BY PERIODIC AND ALMOST PERIODIC FUNCTIONS

BY

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E. Marczewski and C. Ryll-Nardzewski have asked some questions on interpolation by periodic (almost periodic) functions. The general formulation can be made as follows:

A sequence $\{t_n\}$ *of positive numbers is said to have the property* (P) *or* (P') *respectively in a class* K *of sequences of real numbers, if for every* $\{\varepsilon_n\} \epsilon K$ *there exists a continuous periodic, or almost periodic (in the sense of Bohr) respectively, function* $f(t)$ $(-\infty < t < \infty)$ *such that*

$$f(t_n) = \varepsilon_n \quad \textit{for} \quad n = 1, 2, \dots$$

The problem is to find conditions on $\{t_n\}$ implying (P) or (P').

The following results are known: Lipiński [2] has proved that every sequence $\{t_n\}$ for which

$$\frac{t_{n+1}}{t_n} \geqslant \frac{S + u_{n+2}}{u_{n+1}},$$

where $u_n > 0$ $(n = 1, 2, \dots)$ and

$$\sum_{n=1}^{\infty} u_n = S < \infty$$

has the property (P) in the class K of all bounded sequences. Mycielski [3] has proved that every sequence $\{t_n\}$ satisfying the condition

$$t_{n+1} \geqslant (3+\beta) t_n \quad \text{for} \quad n = 1, 2, \dots,$$

where β is any positive constant, has the property (P) in the class K_2 of all sequences taking values 0 or 1. The property (P') in the class of all bounded sequences can be deduced therefrom in view of the main approximation theorem. Ryll-Nardzewski [4] has shown that the sequence $\{3^n\}$ but no sequence with

$$0 < t_n < C \cdot 2^n \quad (C \text{ any constant}, n = 1, 2, \dots)$$

has the property (P) in the class K_2. Later Ryll-Nardzewski proved that the sequence $\{2^n\}$ has the property (P') in the class K of all bounded sequences. Hartman [1] has shown that for every integer $k>0$ the sequence $\{n^k\}$ $(n=1,2,\ldots)$ has not the property (P') in the class K_2.

In this note the last result of Ryll-Nardzewski will be extended (Theorem 2) to all sequences $\{t_n\}$ with

$$t_{n+1} \geqslant (1+\beta)t_n \quad (n=1,2,.;\ \beta>0).$$

We put

$$q_n = \frac{t_{n+1}}{t_n} \quad (n=1,2,\ldots).$$

In the proofs we will use closed intervals $[a_n, b_n]$ satisfying some of the following conditions:

(A_1) $a_1>0$,

(A_{n+1}) $q_n a_n \leqslant a_{n+1} < b_{n+1} \leqslant q_n b_n$,

(B_n) $[a_n, b_n] \subset \begin{cases} [0,\frac{1}{2}] \ (\mathrm{mod}\,1), & \text{if } \varepsilon_n = 0, \\ [\frac{1}{2},1] \ (\mathrm{mod}\,1), & \text{if } \varepsilon_n = 1, \end{cases}$

(C_n) there exists an integer $m>0$ (depending on n) and a positive constant γ (independent of n) such that

$$b_{n+m} \leqslant \frac{t_{n+m}}{t_n}(b_n-\gamma).$$

LEMMA 1. *Every sequence $\{t_n\}$ for which there exists any sequence of intervals $[a_n, b_n]$ satisfying conditions (A_n), (B_n) and (C_n) for $n=1,2,\ldots$ has the property* (P) *in the class K_2.*

Proof. Let us consider the closed intervals

$$\Delta_n = \left[\frac{t}{b_n}, \frac{t_n}{a_n}\right] \quad (n=1,2,\ldots).$$

From (A_{n+1}) it follows that $\Delta_{n+1} \subset \Delta_n$. Hence there exists a number

$$\delta \in \bigcap_{n=1}^{\infty} \Delta_n.$$

Obviously the inequalities

$$a_n \leqslant \frac{t_n}{\delta} \leqslant b_n \tag{1}$$

hold for every n.

Let us fix now an index n and choose the number m so that (C_n) is fulfilled. By (1) and (C_n), we have the inequality

$$\frac{t_{n+m}}{\delta} \leqslant b_{n+m} \leqslant \frac{t_{n+m}}{t_n}(b_n-\gamma),$$

from which it follows that

$$\frac{t_n}{\delta} \leqslant b_n-\gamma.$$

Finally for $n=1,2,\ldots$ we obtain the inequality

$$a_n \leqslant \frac{t_n}{\delta} \leqslant b_n-\gamma,$$

where γ is a positive constant which does not depend on n. From (B_n) we get then

$$\frac{t_n}{\delta} \in \begin{cases} [\frac{1}{2}, 1-\gamma]\,(\mathrm{mod}\,1), & \text{if } \varepsilon_n = 1. \\ [0, \frac{1}{2}-\gamma]\,(\mathrm{mod}\,1), & \text{if } \varepsilon_n = 0, \end{cases}$$

Let $\varphi(t)$ be a continuous function with period 1, such that $\varphi(t)=0$ in $[0,\frac{1}{2}-\gamma]$, $\varphi(t)=1$ in $[\frac{1}{2},1-\gamma]$ and $|\varphi(t)|\leqslant 1$. Evidently the function $f(t)=\varphi(t/\delta)$ satisfies the equations $f(t_n)=\varepsilon_n$ for $n=1,2.\ldots$, i. e., the sequence $\{t_n\}$ has the property (P) in K_2, q. e. d.

THEOREM 1. *Every sequence $\{t_n\}$ satisfying the conditions*

$$q_n \geqslant 1+\alpha \quad (n=1,2,\ldots;\ \alpha>0), \tag{2}$$

$$\text{if } q_n<3, \text{ then } q_{n+1} \geqslant (3+\beta)\frac{2}{q_n-1} \quad (\beta>0), \tag{3}$$

$$\text{if } 3\leqslant q_n \leqslant 3+\beta, \text{ then } q_{n+1} \geqslant 3+\beta, \tag{4}$$

has the property (P) *in K_2.*

In the proof we assume that

$$\alpha \leqslant 2, \tag{5}$$

this restriction being inessential.

The proof of Theorem 1 is based on two lemmas.

LEMMA 2. *Under the assumptions of Theorem* 1 *there exists a sequence of closed intervals* $\{[a_n, b_n]\}$ *satisfying for* $n = 1, 2, \ldots,$ *not only* (A_n) *and* (B_n) *but also the following conditions: if* $q_{n-1} \geqslant 3$, *then*

$$(D_n) \qquad b_n - a_n = \frac{1}{2};$$

if $q_{n-1} < 3$, *then*

$$(E_n) \qquad b_n - a_n \geqslant \tfrac{1}{4}(q_{n-1} - 1),$$

where we put $q_0 = 3$.

Proof. The existence of an interval $[a_1, b_1]$ satisfying (A_1), (B_1) and (D_1), is obvious. Let us suppose that for any $n = k$ the condition (D_k) holds. In this case

$$(6) \qquad d_k = q_k(b_k - a_k) = \tfrac{1}{2} q_k.$$

Since $q_k \geqslant 1 + a$, $a > 0$, there exists a closed interval $[a_{k+1}, b_{k+1}]$ satisfying (A_{k+1}) and (B_{k+1}). Moreover, in case when $q_k < 3$, by (6), the interval $[a_{k+1}, b_{k+1}]$ can be chosen so that the inequality

$$b_{k+1} - a_{k+1} \geqslant \tfrac{1}{2}(d_k - \tfrac{1}{2}) = \tfrac{1}{4}(q_k - 1)$$

holds, which coincides with (E_{k+1}). It is easy to see that in case $q_k \geqslant 3$ the condition (D_{k+1}) can be fulfilled.

Let us suppose now that for $n = k$ we have

$$q_{k-1} < 3 \quad \text{and} \quad b_k - a_k \geqslant \tfrac{1}{4}(q_{k-1} - 1).$$

In this case, by (3), we obtain

$$d_k = q_k(b_k - a_k) \geqslant (3+\beta)\frac{2}{q_{k-1} - 1} \cdot \frac{q_{k-1} - 1}{4} > \frac{3}{2},$$

which implies the existence of an interval $[a_{k+1}, b_{k+1}]$ satisfying (A_{k+1}), (B_{k+1}) and (D_{k+1}). Thus Lemma 2 is proved.

LEMMA 3. *The sequence* $\{b_n\}$ *of Lemma* 2 *can be chosen so that for every* n *the condition* (C_n) *with* $\gamma = \alpha\beta/12(3+\beta)$ *holds.*

Proof. Let us fix an index n. We shall distinguish three cases: (a) $1 + a \leqslant q_n < 3$, (b) $3 \leqslant q_n < 3 + \beta$, (c) $q_n > 3 + \beta$.

In case (a), by (3), we have

$$(7) \qquad q_{n+1} = (3+y)\frac{2}{q_n - 1}$$

with

$$(8) \qquad y \geqslant \beta.$$

Since $q_n < 3$, we obtain from (E_{n+1})

$$b_{n+1} - a_{n+1} \geqslant \tfrac{1}{4}(q_n - 1).$$

Hence, by (7),

$$d_{n+1} = q_{n+1}(b_{n+1} - a_{n+1}) \geqslant \frac{3}{2} + \frac{1}{2}y.$$

Consequently there exists such b_{n+2} as satisfies not only (A_{n+2}), (B_{n+2}), and (D_{n+2}), but also the inequality

$$b_{n+2} \leqslant q_{n+1}b_{n+1} - \frac{y}{2} = q_{n+1}\left(b_{n+1} - \frac{y}{2q_{n+1}}\right)$$

$$= q_{n+1}\left[b_{n+1} - \frac{y(q_n - 1)}{4(3+y)}\right].$$

Hence, in virtue of (8) and (2),

$$(9) \qquad b_{n+2} \leqslant q_{n+1}\left[b_{n+1} - \frac{\alpha\beta}{4(3+\beta)}\right].$$

By (9) and (A_{n+1}), we obtain

$$b_{n+2} \leqslant q_{n+1} q_n \left[b_n - \frac{\alpha\beta}{4q_n(3+\beta)}\right] \leqslant \frac{t_{n+2}}{t_n}\left[b_n - \frac{\alpha\beta}{12(3+\beta)}\right],$$

which coincides with (C_n) with $m = 2$.

In case (b) one has, by (4),

$$q_{n+1} \geqslant 3 + \beta.$$

Let us put

$$(10) \qquad q_n = 3 + z \quad (z \geqslant 0),$$

$$(11) \qquad q_{n+1} = 3 + u \quad (u \geqslant \beta).$$

In quite a similar way as in case (a) we can show that b_{n+1} and b_{n+2} can be chosen so that besides the conditions (A_{n+1}), (B_{n+1}), (D_{n+1}), (A_{n+2}), (B_{n+2}), (D_{n+2}) the following inequalities would be true:

$$(12) \qquad b_{n+1} \leqslant q_n\left[b_n - \frac{z}{2(3+z)}\right],$$

$$(13) \qquad b_{n+2} \leqslant q_{n+1}\left[b_{n+1} - \frac{u}{2(3+u)}\right].$$

(In the proof of (12) it must be taken into consideration that $q_{n-1} \geqslant 3+\beta$ and so we have (D_n).) By (13), (12), and (10), we obtain

$$b_{n+2} \leqslant q_{n+1} q_n \left[b_n - \frac{z}{2(3+z)} - \frac{u}{2(3+z)(3+u)} \right]$$

$$= \frac{t_{n+2}}{t_n} \left[b_n - \frac{1}{2} + \frac{1}{3+z} + \frac{3}{2(3+z)(3+u)} \right].$$

Hence, by (5), (10) and (11),

$$b_{n+2} \leqslant \frac{t_{n+2}}{t_n} \left[b_n - \frac{1}{2} + \frac{1}{3} + \frac{1}{2(3+\beta)} \right]$$

$$= \frac{t_{n+2}}{t_n} \left[b_n - \frac{\beta}{6(3+\beta)} \right] \leqslant \frac{t_{n+2}}{t_n} \left[b_n - \frac{a\beta}{12(3+\beta)} \right],$$

i. e., in this case the condition (C_n) for $m = 2$ is also satisfied.

In case (c) we apply the inequality (9) or (13), respectively, according to whether $q_{n-1} < 3$ or $q_{n-1} \geqslant 3$. Replacing in both of them n by $n-1$ we get (C_n) with $m = 1$. Thus Lemma 3 is proved.

Theorem 1 is now a direct consequence of Lemmas 1 and 3.

THEOREM 2. *Every sequence* $\{t_n\}$ *for which*

(14) $$t_{n+1} \geqslant (1+a) t_n \quad (n = 1, 2, \ldots;\ a > 0)$$

has the property (P′) *in the class* K *of all bounded sequences.*

Proof. Obviously we may suppose (5) without loss of generality. Let us denote by s a positive integer such that

(15) $$(1+a)^{s-3} < \frac{7}{a} \leqslant (1+a)^{s-2}.$$

We shall choose a subsequence $\{t_{n_i}\}$ satisfying the conditions

(16) $$t_{n_1} = t_1, \quad t_{n_{i+1}-1} < (1+a)^s t_{n_i} \leqslant t_{n_{i+1}}.$$

The sets

$$U_k = \bigcup_{i=1}^{\infty} \{t_n\colon (1+a)^{k-1} t_{n_i} \leqslant t_n < (1+a)^k t_{n_i}\} \quad (k = 1, \ldots, s)$$

are disjoint. Hence every t_n belongs to one and only one of them.

Let us fix now an index k. We shall prove that if $t', t'' \in U_k$, $t' < t''$, then

(17) $$t'' \geqslant 7 \frac{1+a}{a} t'.$$

We note that t' and t'' cannot satisfy the inequality

$$(1+a)^{k-1} t_{n_i} \leqslant t' < t'' < (1+a)^k t_{n_i}.$$

In fact, from this inequality it follows that

$$t' < t'' < (1+a) t',$$

which contradicts (14). Consequently there exists an index i such that

(18) $$t' < (1+a)^k t_{n_i},$$

(19) $$t'' \geqslant (1+a)^{k-1} t_{n_{i+1}}.$$

By (19), (16), (18) and (15), we have (17), since

$$t'' \geqslant (1+a)^{k-1} t_{n_{i+1}} \geqslant (1+a)^{s-1} (1+a)^k t_{n_i} > (1+a)^{s-1} t' \geqslant 7 \frac{1+a}{a} t'.$$

As a consequence of (17) and (5) the inequality

(20) $$\frac{t''}{t'} \geqslant 7 \frac{1+a}{a} \geqslant \frac{21}{2}$$

holds for every $t', t'' \in U_k$, $t'' > t'$.

Now let us fix the indices k and l ($k \neq l$) and consider the subsequence of $\{t_n\}$ consisting of all the elements x_n ($n = 1, 2, \ldots$) belonging to the set $U_k \cup U_l$. We shall show that the subsequence $\{x_n\}$ fulfills the assumptions of Theorem 1. For this purpose it is enough to prove that conditions (3) and (4) are satisfied for $\beta = 0{,}1$.

1° If both elements x_n and x_{n+1} belong to one of the sets U_k or U_l, then we have, by (20),

$$q_n = \frac{x_{n+1}}{x_n} > 3{,}1$$

and there is nothing more to prove.

2° Let us assume that $x_n \in U_k$, $x_{n+1}, x_{n+2} \in U_l$. In this case, by (20), we have

(21) $$q_{n+1} = \frac{x_{n+2}}{x_{n+1}} \geqslant 7 \frac{1+a}{a} \geqslant \frac{21}{2}.$$

Hence, assuming that $q_n \geqslant 3$, we see that (4) is satisfied. So is (3) in virtue of (14) and (21) since putting $q_n < 3$ we have

$$3{,}1 \cdot \frac{2}{q_n - 1} \leqslant 3{,}1 \cdot \frac{2}{a} < \frac{7}{a} < q_{n+1}.$$

There is still one case to be considered:

3° $x_n, x_{n+2} \in U_k$, $x_{n+1} \in U_l$. By (21) and (14),

$$q_{n+1}(q_n - 1) = q_{n+1} q_n \left(1 - \frac{1}{q_n}\right) = \frac{x_{n+2}}{x_n}\left(1 - \frac{1}{q_n}\right)$$
$$\geqslant 7\,\frac{1+a}{a}\left(1 - \frac{1}{1+a}\right) = 7.$$

Hence

$$q_{n+1} \geqslant \frac{7}{q_n - 1} > 3{,}1 \cdot \frac{2}{q_n - 1}.$$

Therefore (3) is fulfilled. Supposing that $3 \leqslant q_n < 3{,}1$, by (20), we obtain

$$q_{n+1} = \frac{q_n q_{n+1}}{q_n} = \frac{x_{n+2}}{x_n}\cdot\frac{1}{q_n} \geqslant \frac{10{,}5}{3{,}1} > 3{,}1$$

and so (4) is also fulfilled.

Since the sequence $\{x_n\}$ satisfies the assumptions of Theorem 1, it has the property (P) in the class K_2. Hence there exists a continuous periodic function $g_{k,l}(t)$ such that

$$g_{k,l}(t) = \begin{cases} 1, & \text{if} \quad t \in U_k, \\ 0, & \text{if} \quad t \in U_l, \end{cases}$$
$$|g_{k,l}(t)| \leqslant 1.$$

Therefore the function

$$g_k(t) = \prod_{\substack{l=1 \\ l \neq k}}^{s} g_{k,l}(t)$$

is an almost periodic function in the sense of Bohr satisfying for $n = 1, 2, \ldots$ the conditions

$$(22) \qquad g_k(t_n) = \begin{cases} 1, & \text{if} \quad t_n \in U_k \\ 0, & \text{if} \quad t_n \notin U_k, \end{cases} \qquad |g_k(t)| \leqslant 1.$$

Taking into account (20), by the Theorem of Mycielski [3], each of the sets U_k ($k = 1, 2, \ldots, s$) has the property (P) in the class K_2. Thus, given a sequence $\{\varepsilon_n\}$ with $\varepsilon_n = 0$ or 1, choose a continuous periodic function $f_k(t)$ satisfying

$$f_k(t_n) = \varepsilon_n, \quad \text{if} \quad t_n \in U_k,$$
$$|f_k(t)| \leqslant 1.$$

Hence, according to (22) the function

$$f(t) = \sum_{k=1}^{s} g_k(t)\cdot f_k(t)$$

takes the values ε_n in t_n. This shows that the sequence $\{t_n\}$ has the property (P′) in the class K_2. Consequently, by the main approximation theorem for almost periodic functions, the sequence $\{t_n\}$ has the property (P′) in the class K of all bounded sequences, q. e. d.

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Reçu par la Rédaction le 6. 2. 1963