Now

$$\sum_{k\leqslant x} g_s(k) = \sum_{k\leqslant x}\sum_{d|k} g(d)\tau_s(k/d)$$

$$= \sum_{d\leqslant x} g(d) \sum_{\substack{k\leqslant x\\ k|d}} \tau_s(k/d) = \sum_{d\leqslant x} \frac{g(d)}{d} \sum_{l\leqslant x/d} d\tau_s(l).$$

Let us put

$$a_{n,k} = \frac{1}{k\log^{s-1}k} \sum_{l\leqslant k/n} n\tau_s(l)$$

if $k\geqslant n$, and $a_{n,k}=0$ if $k<n$. Moreover, let $c_n = g(n)/n$. Condition (ii) follows from (*), and (i) follows from

$$|a_{n,k}| = \frac{1}{k\log^{s-1}k}\sum_{l\leqslant k/n} n\tau_s(l)$$

$$= \frac{n}{k\log^{s-1}k}\left(\frac{1}{(s-1)!}\,\frac{k}{n}\log^{s-1}\left(\frac{k}{n}\right) + o\left(\frac{k}{n}\log^{s-1}\frac{k}{n}\right)\right)$$

$$= \frac{1}{(s-1)!}\left(1-\frac{\log n}{\log k}\right)^{s-1} + o\left(\left(1-\frac{\log n}{\log k}\right)^{s-1}\right) = \frac{1}{(s-1)!} + O(1) = O(1)$$

in the case $k\geqslant n$ and is obvious in the case $k<n$. As $a_n = 1/(s-1)!$ for all n, the theorem follows.

It should be remarked that the same method leads to a similar theorem (in case $s=1$) regarding the unitary convolution (see [1]), namely:

If the series $\sum_{n=1}^{\infty} g(n)/n$ *is absolutely convergent and* $f(n) = \sum_{\substack{d|n\\(d,n/d)=1}} g(d)$, *then*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x} f(n) = \sum_{n=1}^{\infty}\frac{g(n)\varphi(n)}{n^2}.$$

REFERENCES

[1] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Mathematische Zeitschrift 74 (1960), p. 66-80.

[2] — *Arithmetical notes I, On a theorem of van der Corput*, Proceedings of the American Mathematical Society 12 (1961), p. 214-217.

MATHEMATICAL INSTITUTE, WROCŁAW UNIVERSITY

Reçu par la Rédaction le 26. 11. 1962

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

A SIMPLE REMARK ON MATRICES

BY

A. GOETZ (WROCŁAW)

This paper deals with a certain representation of matrices which can be useful as a tool for treating the projective group. The author is sure that the results are known to many persons, but he doubts if they have ever been published.

1. Consider the group $GL(n+1, K)$ of matrices of the form

$$A = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{pmatrix}$$

over any field K of characteristic 0.

It is a matter of elementary calculations to notice that the set G of matrices with constant sums of rows, i. e. satisfying the condition

$$\sum_{i=0}^{n} a_{ij} = \lambda \text{ does not depend on } j,$$

as well as the set G_0 of matrices satisfying the condition

$$\sum_{i=0}^{n} a_{ij} = 1$$

are subgroups of $GL(n+1)$.

The subgroup G_0 is isomorphic with the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

The isomorphism, being defined as

$$\begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \dots & 0 \\ a_{10} & a_{11}-a_{10} & \dots & a_{1n}-a_{10} \\ \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1}-a_{n0} & \dots & a_{nn}-a_{n0} \end{pmatrix},$$

can be also obtained by the inner authomorphism

$$A \to \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} A \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Denote by H the group of matrices of the form

$$\begin{pmatrix} \mu_0 & 0 & \dots & 0 \\ 0 & \mu_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}$$

and by H_0 the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

2. I call a matrix $A \in GL(n+1)$ *exceptional*, if the sum of elements of some row is 0. Of course this cannot happen for all rows of a non-degenerated matrix. It is also evident that a matrix sufficiently near to the matrix λE, where $\lambda \in K$ and E is the unity matrix, is non-exceptional.

THEOREM 1. *Every non-exceptional matrix A can be uniquely represented in the form*

$$A = aBC,$$

where $a \in K$, $B \in G_0$, $C \in H_0$.

Proof. Write

$$\mu_j = \sum_{i=0}^{n} a_{ij}.$$

Since $\mu_j \neq 0$ $(j = 0, \dots, n)$ by supposition, we can set $b_{ij} = a_{ij}/\mu_j$. Thus $\sum_{i=0}^{n} b_{ij} = 1$ and the matrix

$$B = \begin{pmatrix} b_{00} & \dots & b_{0n} \\ \dots & \dots & \dots \\ b_{n0} & \dots & b_{nn} \end{pmatrix}$$

belongs to G_0. Set now $a_i = \mu_0$ and $\lambda_i = \mu_i/\mu_0$; then

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \in H_0,$$

and an easy calculation shows that

$$A = aB \cdot C.$$

On the other hand, if such a representation of A is given, then necessarily

$$a = \sum_{i=0}^{n} a_{i0} \quad \text{and} \quad a\lambda_j = \sum_{i=0}^{n} a_{ij},$$

hence the representation is unique.

THEOREM 2. *Every matrix A is a product of two non-exceptional matrices $A = A_1 A_2$. A_1 can be chosen in the group H (or H_0), but even under the last restriction the decomposition is not unique.*

Proof. Notice first that for each fixed j the solutions of the equation

$$\sum_{i=0}^{n} a_{ij} x_i = 0$$

lie in an n-dimensional subspace of K^{n+1}. Therefore there exists a vector $(\lambda_0, \lambda_1, \dots, \lambda_n)$ such that

$$\sum_{i=0}^{n} a_{ij}\lambda_i \neq 0 \quad \text{and} \quad \lambda_j \neq 0$$

for every $j = 0, 1, \dots, n$. (In fact every vector has this property, except of vectors lying in the union of $2n+2$ n-dimensional subspaces of K^{n+1}). Set now

$$A_1 = \begin{pmatrix} 1/\lambda_0 & 0 & 0 & \dots & 0 \\ 0 & 1/\lambda_1 & 0 & \dots & 0 \\ 0 & 0 & 1/\lambda_2 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1/\lambda_n \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \lambda_0 a_{00} & \lambda_0 a_{01} & \dots & \lambda_0 a_{0n} \\ \lambda_1 a_{10} & \lambda_1 a_{11} & \dots & \lambda_1 a_{1n} \\ \dots & \dots & \dots & \dots \\ \lambda_n a_{n0} & \lambda_n a_{n1} & \dots & \lambda_n a_{nn} \end{pmatrix}.$$

$A_1 \epsilon H$ and A_2 are both non-exceptional matrices and $A = A_1 A_2$, q. e. d.

MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 6. 2. 1963

COLLOQUIUM MATHEMATICUM

VOL. XI 1963 FASC. 1

ON A PROBLEM OF INTERPOLATION BY PERIODIC AND ALMOST PERIODIC FUNCTIONS

BY

E. STRZELECKI (WROCŁAW)

E. Marczewski and C. Ryll-Nardzewski have asked some questions on interpolation by periodic (almost periodic) functions. The general formulation can be made as follows:

A sequence $\{t_n\}$ of positive numbers is said to have the property (P) *or* (P′) *respectively in a class K of sequences of real numbers, if for every $\{\varepsilon_n\} \epsilon K$ there exists a continuous periodic, or almost periodic (in the sense of Bohr) respectively, function $f(t)$ $(-\infty < t < \infty)$ such that*

$$f(t_n) = \varepsilon_n \quad \textit{for} \quad n = 1, 2, \dots$$

The problem is to find conditions on $\{t_n\}$ implying (P) or (P′).

The following results are known: Lipiński [2] has proved that every sequence $\{t_n\}$ for which

$$\frac{t_{n+1}}{t_n} \geqslant \frac{S + u_{n+2}}{u_{n+1}},$$

where $u_n > 0$ $(n = 1, 2, \dots)$ and

$$\sum_{n=1}^{\infty} u_n = S < \infty$$

has the property (P) in the class K of all bounded sequences. Mycielski [3] has proved that every sequence $\{t_n\}$ satisfying the condition

$$t_{n+1} \geqslant (3+\beta) t_n \quad \text{for} \quad n = 1, 2, \dots,$$

where β is any positive constant, has the property (P) in the class K_2 of all sequences taking values 0 or 1. The property (P′) in the class of all bounded sequences can be deduced therefrom in view of the main approximation theorem. Ryll-Nardzewski [4] has shown that the sequence $\{3^n\}$ but no sequence with

$$0 < t_n < C \cdot 2^n \quad (C \text{ any constant}, \ n = 1, 2, \dots)$$