of condensation of the set

$$\bigcup_{j\geqslant 1}\bigcap_{r\geqslant j} A_{q_r},$$

and every open set containing a point of S also contains a perfect subset of $A_{q_j}\cap A_{q_{j+1}}\cap\ldots$ for some j.

Proof. It is clear how nearly all the steps in the proof of Theorem 1 have to be modified to provide a proof of Theorem 2; the only difficulty is in the choice of the disjoint closed subsets H_0 and H_1 and the subsequent choice of the subsets (1) for $k = 2, 3, \ldots$ These choices are justified by the following lemma, which we prove by using one of the ideas we have already used:

LEMMA. *Under the conditions of Theorem 2, if A is a μ-measurable set with $\mu(A) > 0$, we can choose two disjoint closed subsets H_0 and H_1 of A with $\mu(H_0) > 0$, $\mu(H_1) > 0$.*

Proof. As A is μ-measurable and $\mu(A) > 0$, we can choose a closed set B contained in A with $\mu(B) > 0$. Let $X_1, X_2, \ldots$ be a countable base for the open sets of X. Take

$$C = B - \bigcup{}' X_r,$$

the union being taken over all the integers r for which $\mu(B\cap X_r) = 0$. Then C is closed and

$$\mu(C) = \mu(B) - \sum_{\mu(B\cap X_r)=0} \mu(B\cap X_r) = \mu(B) > 0.$$

Hence C contains at least one point, c say. As $\mu\big((c)\big) = 0$, we can choose an open set G with $c\in G$ and $\mu(G) < \mu(C)$. Choose r so that $c\in X_r$ and $X_r\subset G$. Then, as $c\in X_r$, we have $\mu(B\cap X_r) > 0$, so that

$$\mu(C\cap G) \geqslant \mu(B\cap X_r) > 0.$$

Finally, take H_0 to be a closed subset of $C\cap G$ with $\mu(H_0) > 0$, and take $H_1 = C\cap(X-G)$. It is easy to verify that these sets satisfy our requirements.

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ON A COMBINATORICAL PROBLEM OF K. ZARANKIEWICZ

BY

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Zarankiewicz [6] raised the following problem. Let A_n be a square matrix of order n, consisting exclusively of 1's and 0's; j is a positive integer with $2\leqslant j < n$. The problem consists in finding the smallest number of 1's still assuring the existence of a minor of order j, consisting exclusively of 1's. Let us denote this number by $k_j(n)$.

I. Reiman in [5] solves this problem for $j = 2$ and proves that

$$k_2(n) \leqslant \tfrac{1}{2}(n+n\sqrt{4n-3})+1. \tag{1}$$

Hyltén-Cavallius [3] proves the inequality

$$k_j(n) < 1+(j-1)n+[(j-1)^{1/j}n^{(2j-1)/j}], \tag{2}$$

where $[a]$ is the integer part of a.

This paper deals with improvement of this result. We prove namely that

$$k_j(n) < 1+\left[\frac{j-1}{2}n+(j-1)^{1/j}n^{(2j-1)/j}\right] \tag{3}$$

which is somewhat better than (2), e. g. (2) gives $k_3(8) < 56$ and (3) implies $k_3(8) < 48$. However, (3) is worse than (1) for $j = 2$.

Let k_i denote the number of 1's in the i-th row of A_n. It is obviously sufficient to deal with matrices with

$$k_1\geqslant k_2\geqslant\ldots\geqslant k_n\geqslant j-1. \tag{4}$$

To prove (3) we need three lemmas.

LEMMA 1. *For an arbitrary integer $n > 0$ and any real a_i, b_i ($i = 1, 2, \ldots, n$) with $a_1\geqslant a_2\geqslant\ldots\geqslant a_n$ and $b_1\geqslant b_2\geqslant\ldots\geqslant b_n$ we have*

$$n\sum_{i=1}^{n} a_i b_i \geqslant \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j$$

(see e. g. [2], p. 43, theorem 43).

LEMMA 2. *If* $k = \frac{1}{n}\sum_{i=1}^{n} k_i$, *then*

$$\sum_{i=1}^{n}\binom{k_i}{j} \geqslant n\binom{K}{j} \tag{5}$$

for any positive integer $j < n$.

Proof. We proceed by induction with respect to j. For $j = 1$ formula (5) is clearly satisfied. So let us suppose that it holds for a number $h < n-1$. According to (4)

$$\binom{k_1}{h} \geqslant \binom{k_2}{h} \geqslant \ldots \geqslant \binom{k_n}{h}, \qquad \frac{k_1-h}{h+1} \geqslant \frac{k_2-h}{h+1} \geqslant \ldots \geqslant \frac{k_n-h}{h+1}.$$

In virtue of lemma 1 we get therefore by the induction hypothesis

$$\sum_{i=1}^{n}\binom{k_i}{h+1} = \sum_{i=1}^{n}\binom{k_i}{h}\frac{k_i-h}{h+1} \geqslant \frac{1}{n}\sum_{i=1}^{n}\binom{k_i}{h}\sum_{i=1}^{n}\frac{k_i-h}{h+1}$$

$$= \left\{\sum_{i=1}^{n}\binom{k_i}{h}\right\}\frac{K-h}{h+1} \geqslant n\binom{K}{h}\frac{K-h}{h+1} = n\binom{K}{h+1},$$

q. e. d.

LEMMA 3. *If* $U = \frac{1}{2}(j-1)+(j-1)^{1/j}n^{(j-1)/j}$, *then*

$$n\binom{U}{j} > (j-1)\binom{n}{j} \tag{6}$$

for any integer j *with* $2 \leqslant j < n$.

Proof. We shall distinguish two cases.

1. If j is even, then (6) can be written in the form

$$(n^{1/j}U)\big(n^{1/j}(U-1)\big)\ldots\big(n^{1/j}(U-\tfrac{1}{2}(j-2))\big)\big(n^{1/j}(U-\tfrac{1}{2}j)\big)\ldots\big(n^{1/j}(U-j+1)\big)$$

$$> \big((j-1)^{1/j}n\big)\ldots\big((j-1)^{1/j}(n-\tfrac{1}{2}(j-2))\big)\big((j-1)^{1/j}(n-\tfrac{1}{2}j)\big)\ldots$$

$$\ldots\big((j-1)^{1/j}(n-j+1)\big),$$

and, with some modifications,

$$\big(n^{2/j}U(U-j+1)\big)\big((n^{2/j}(U-1)(U-j+2)\ldots(n^{2/j}(U-\tfrac{1}{2}(j-2))(U-\tfrac{1}{2}j)\big)$$

$$> \big((j-1)^{2/j}n(n-j+1)\big)\big((j-1)^{2/j}(n-1)(n-j+2)\big)\ldots$$

$$\ldots\big((j-1)^{2/j}(n-\tfrac{1}{2}(j-2))(n-\tfrac{1}{2}j)\big).$$

The condition $j < n$ implies $U > j-1$; therefore all factors on both sides are positive. Hence it is sufficient to prove that the r-th factor on the left-hand side is larger that the r-th factor on the right-hand side where $1 \leqslant r \leqslant j/2$ since the number of factors is $j/2$, i. e. that

$$n^{2/j}(U-r+1)(U-j+r) > (j-1)^{2/j}(n-r+1)(n-j+r).$$

This means that

$$n^{2/j}\big((j-1)^{1/j}n^{(j-1)/j}+\tfrac{1}{2}(j-1)-r+1\big)\big((j-1)^{1/j}n^{(j-1)/j}-\tfrac{1}{2}(j-1)+r-1\big)$$

$$> (j-1)^{2/j}(n-r+1)(n-j+r),$$

and with some modifications

$$(j-1)^{(j+2)/j}n > \big(\tfrac{1}{2}(j-1)-r+1\big)^2 n^{2/j}+(r-1)(j-r)(j-1)^{2/j}. \tag{7}$$

Since $r \leqslant j/2$, we have

$$(r-1)(j-r)(j-1)^{2/j} < \tfrac{1}{2}(j-1)^2(j-1)^{2/j} < \tfrac{1}{2}(j-1)^2 n^{2/j}. \tag{8}$$

Since $2 \leqslant j < n$, we have $n^{(j-2)/j} \geqslant j^{(j-2)/j}$. Multiplying this inequality by $n^{2/j}(j-1)^{(2+j)/j}$, we get

$$(j-1)^{(2+j)/j}n \geqslant j^{(j-2)/j}(j-1)^{(j+2)/j}n^{2/j} \geqslant (j-1)^2 n^{2/j}$$

$$> \frac{(j-1)^2}{4}n^{2/j}+\frac{(j-1)^2}{2}n^{2/j} \geqslant \left(\frac{j-1}{2}-r+1\right)^2 n^{2/j}+\frac{(j-1)^2}{2}n^{2/j}.$$

Hence we infer (7) using (8).

2. If j is odd, the proof is analogous.

Now, to prove (3) observe that if

$$\sum_{i=1}^{n}k_i > \frac{j-1}{2}n+(j-1)^{1/j}n^{(2j-1)/j} = nU,$$

then

$$\sum_{i=1}^{n}\binom{k_i}{j} > (j-1)\binom{n}{j}. \tag{9}$$

In fact, since $K > U > j-1$, we have $\binom{K}{j} > \binom{U}{j}$. Hence, by lemma 2 and 3,

$$\sum_{i=1}^{n}\binom{k_i}{j} \geqslant n\binom{K}{j} > n\binom{U}{j} > (j-1)\binom{n}{j}.$$

According to the result of [1], relation (9) is a sufficient condition for the existence of a minor of order j, consisting exclusively of 1's. Thus (3) is proved.

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ON A SUMMATION FORMULA OF E. COHEN

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The following theorem is well known:

If the series $\sum_{n=1}^{\infty} g(n)/n$ *is absolutely convergent and* $f(n) = \sum_{d|n} g(d)$, *then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n) = \sum_{n=1}^{\infty} g(n)/n.$$

Recently E. Cohen [2] proved the following generalization of this theorem:

If the series $\sum_{n=1}^{\infty} g(n)/n$ *is absolutely convergent and* $g_s(n) = \sum_{d|n} g(d)\tau_s(n/d)$ (*where* $\tau_s(n)$ *is defined by* $\tau_1(n) = 1$, $\tau_{s+1}(n) = \sum_{d|n}\tau_s(d)$), *then*

$$\lim_{x\to\infty} \frac{1}{x\log^{s-1}x} \sum_{n\leqslant x} g_s(n) = \frac{1}{(s-1)!} \sum_{n=1}^{\infty} \frac{g(n)}{n} \qquad (s = 1, 2, \ldots).$$

In this note we give a simple proof of the theorem of E. Cohen, based on the remark that if $\|a_{n,k}\|$ is an infinite matrix satisfying the conditions

(i) $|a_{n,k}| \leqslant M$ with some M independent of k and n,

(ii) for every n the sequence $\{a_{n,k}\}_{k=1}^{\infty}$ is convergent to, say, a_n,

then from $\sum_{m=1}^{\infty} |c_m| < \infty$ follows

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{n,k} c_n = \sum_{n=1}^{\infty} a_n c_n.$$

The following formula is well-known and can be easily proved by induction:

$$(*) \qquad \lim_{x\to\infty} \frac{1}{x\log^{s-1}x} \sum_{m\leqslant x} \tau_s(m) = 1/(s-1)!.$$