But $p > 2$, and consequently $q < 2 < p$. Hence $L^p(G) \subseteq L^p(\mathfrak{g})$, and consequently $L^p(G) = L^p(\mathfrak{g}) = L^p(\mathfrak{g})$. Therefore $L^p(G)$ is an algebra under convolution.

Hence $G$ is finite.

Note. The author has proved after submitting this paper that for any locally compact group $G$ the space $L^p(G)$ is closed for convolution for some $p > 2$ if $G$ is compact.

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The convolution is defined as

\[ x \ast y(t) = \int x(\tau^{-1})y(\tau) \mu(\, d\tau). \]

The following lemma reduces our problem to the case when \( G \) is a unimodular group.

**Lemma 1.** If \( G \) is not unimodular locally compact group, then \( L_p(G) \) is not an algebra under the convolution for any \( p > 1 \).

**Proof.** By our assumption there exists in \( G \) such a \( t_0 \) that \( \Delta(t_0) \neq 1 \). Let \( \mathcal{G}(t_0) \) be a subgroup of \( G \) generated by \( t_0 \). It is the intersection of all closed subgroups of \( G \) containing \( t_0 \). We have either \( \Delta(t_0) > 1 \), or \( \Delta(t_0') > 1 \), so \( \mathcal{G}(t_0) \) is not compact, since the continuous modular function is unbounded on the sequence \( \{t_n\} \), \( n = 0, \pm 1, \pm 2, \ldots \), contained in \( \mathcal{G}(t_0) \). Consequently \( \mathcal{G}(t_0) \) is discrete and consists exactly of all positive and negative powers of \( t_0 \) (cf. [4], lemma 3, or [1]). But in this case the proof of our conclusion given in [3] holds; so does the proof of the theorem 2, section 2, pp. 117 and 118, q. e. d.

**Lemma 2.** If the group \( G \) is not compact then for every compact subset \( A \subset G \) there exists a sequence of elements \( t_nA_n \), \( n = 1, 2, \ldots \), such that

\[ t_nA \cap t_kA = \emptyset \quad \text{for} \quad n \neq k. \]

**Proof.** Take an arbitrary element as \( t_1 \), choose \( t_2 \) in such a way that \( t_1A \cap t_2A = \emptyset \), then choose \( t_3 \) in such a way that \( t_2A \cap t_3A = \emptyset \) and so on. If the \( n \)-th step is impossible, then for any \( t \in G \) there exists a \( t_nA \cap tA = \emptyset \), which is equivalent to \( t \in \mathcal{G}(t_nA) \). But in this case \( G \) would be covered by the finite family of compact sets, which is impossible, q. e. d.

**Lemma 3.** If \( L_p(G) \) is a Banach algebra under the convolution, and if \( p > 2 \), then \( G \) is compact.

**Proof.** By lemma 1 we may assume that \( G \) is a unimodular group. It is to be proved that \( \mu(G) < \infty \) which is equivalent to its compactness. Suppose then that \( \mu(G) = \infty \) and the proof will be given if we get a contradiction. Let \( x(t) \in L_p(G) \), so \( \mathcal{N}(t) = x(t^{-1}) \ast x(t) \). Let \( t \in G \), where \( 1/p + 1/q = 1 \), and consider the functional \( (x, x) \). It is continuous and bounded, generated on \( L_p(G) \) by the function \( x \), taken at the point \( t \ast y \), \( y \in L_p(G) \). For fixed \( x \in L_p \), \( x \ast x \) is a continuous linear functional defined on \( L_p \). So there exists a \( \omega \in L_p \) such that \( (x, x) = \langle \omega \rangle \). In the same way as in [2] we shall show that \( \omega = x \ast x \). In fact,

\[ (x, x) = \int x(t) \overline{x}(t^{-1}) y(t) \mu(\, d\tau). \]

So for every \( x \in L_p(G) \), \( x \in L_p(G) \) we have \( x \ast x \in L_p(G) \). To prove our lemma it is sufficient to construct such an \( x \in L_p \), \( x \in L_p \), that \( x \ast x \) is not in \( L_p \). Let \( U \) be a compact symmetric neighbourhood of the unit \( e \in G \). \( U^p \) is also compact, so by our assumption and by lemma 3 we can choose such sequence \( (t_n) \) of elements of \( G \) that (3) holds with \( U^p \) instead of \( A \). It is clear that for this sequence (3) also holds if we take \( U \) instead of \( A \). We put now

\[ x(t) = \sum_{n=1}^{\infty} a_n \mathcal{N}(t_n^p)^{\alpha}(t), \]

\[ y(t) = \sum_{n=1}^{\infty} b_n \mathcal{N}(t_n^p)^{\beta}(t), \]

\[ z(t) = \chi(t), \]

where \( a_n \) is a sequence of positive reals such that

\[ \sum_{n=1}^{\infty} a_n^p \leq \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n^p = \infty. \]

This is possible because \( q < p \) for \( p > 2 \); \( \chi(t) \) denotes the characteristic function of the set \( A \), i.e.

\[ \chi(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases} \]

It is clear that \( x, y \in L_p \), \( z \in L_p \), and \( x \) and \( y \) are not members of \( L_p \). We shall see that \( x \ast x \in L_p \). Let \( t \in G \). Then \( t \in U \) and \( x \ast x \in L_p \). In fact, consider the convolution

\[ \chi(t) = \int_{G} \mu(U) dx(t). \]

Let \( t \in U \). Then \( t \in U \) and \( \chi(t) = \mu(U) \) if \( t \in G \).

Consequently

\[ \chi(t) = \mu(U) \chi(t) \]

for every \( t \in G \) and \( \chi(t) = \mu(U) \chi(t) \). But \( \|x\| = \infty \), and \( \mu(U) > 0 \), so \( \|x \ast x\| = \infty \) and \( x \ast x \) is not in \( L_p \), q. e. d.
such that $\lim ||x_\alpha||_p = 0$, and $\|f \ast x_\alpha\|_p \geq C$, $n = 1, 2, \ldots$ Taking $x_\alpha(t)$ instead of $x_\alpha(t)$ we have also $\lim ||x_\alpha(t)||_p = 0$ and $\|f \ast x_\alpha\|_p \geq C$, so we may assume that $x_\alpha(t) \geq 0$. Taking a suitable sequence of positive scalars $a_n$ we obtain $\lim ||a_n x_\alpha||_p = 0$, and $\lim ||f \ast a_n x_\alpha||_p = \infty$, so by passing, if necessary, to a subsequence we may assume that

$$||x_\alpha||_p \leq 1/2^n \quad \text{and} \quad ||f \ast x_\alpha||_p \geq n$$

for $n = 1, 2, \ldots$ Now let $y = \sum x_\alpha$; we have $y \in L_\infty$, so $||f \ast y||_p < \infty$. On the other hand, $y \geq x_\alpha$ and so $f \ast y \geq f \ast x_\alpha > 0$. Consequently $||f \ast y||_p \geq ||f \ast x_\alpha||_p \geq n$, which is the contradiction mentioned above, q. e. d.

**Corollary.** If $L_\infty(G)$, $p \geq 1$, is an algebra under the convolution, then it is a Banach algebra (i.e., there exists a submultiplicative norm equivalent to the norm $||\cdot||_p$).

We may formulate now our main result

**Theorem 1.** Let $G$ be a locally compact group and $p > 2$; then the space $L_p(G)$ is an algebra under the convolution if and only if the group $G$ is compact.

We may rewrite also the main result of [3] in the following form:

**Theorem 2.** Let $G$ be a locally compact Abelian group and $p > 1$; then the space $L_p(G)$ is an algebra under the convolution if and only if the group $G$ is compact.

The following problem is open:

P 392. Is the conclusion of theorem 1 true for $1 < p < 2$?

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**ON DECOMPOSITION OF A COMMUTATIVE p-NORMED ALGEBRA INTO A DIRECT SUM OF IDEALS**

by W. ŻELAZKO (WARSAW)

1. In the theory of commutative complex Banach algebras it is known that a Banach algebra $A$ is decomposable into a direct sum of its two non-trivial ideals

$$A = I_1 \oplus I_2,$$

if and only if the compact space $\mathcal{M}$ of all multiplicative linear functionals of $A$ may be written in the form

$$\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2,$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint closed subsets of $\mathcal{M}$.

The decompositions (1.1) and (1.2) are equivalent to the decomposition of the unit $e \in A$ into a sum of two non-zero idempotents

$$e = e_1 + e_2,$$

where

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0.$$

When we have the decomposition (1.3) with (1.4) the decompositions (1.1) and (1.2) may be written by means of the formulas

$$I_1 = e_1 A, \quad I_2 = e_2 A,$$

and

$$\mathcal{M}_1 = \{f \in \mathcal{M}; f(e_1) = 1\}, \quad \mathcal{M}_2 = \{f \in \mathcal{M}; f(e_2) = 1\}.$$

This result was obtained by Šilov [4], who used analytic functions of several variables of elements of $A$. Here is presented a similar result for the class of $p$-normed algebras.

2. A $p$-normed algebra $A$ is a metric algebra complete in the norm $||\cdot||_p$ satisfying

$$||xy||_p \leq ||x||_p ||y||_p, \quad ||x^n||_p = ||x||_p^n.$$