there exists a compact set $X \subset h(S_n - D_h)$ with $0 < \dim X$. So $\dim X \leqslant \dim g(X)$ by the same Hurewicz theorem, and $g(X) \subset Y$ by (ii). We get $0 < \dim Y$.

Questions. We have shown by the example (see p. 46) that for some mapping f which lowers the dimension of S_n the set D_f can be dense in S_n. Then $\dim(S_n - D_f) \leqslant n-1$ (see [1], p. 353). Actually, the set $S_n - D_f$ has the dimension equal to $n-1$. This suggests the following question:

P 390. *Is it true that* $\dim f(S_n) \leqslant n-1$ *implies* $n-1 \leqslant \dim(S_n - D_f)$ *for every mapping* f *of the sphere* S_n $(n = 3, 4, \ldots)$?

The proposition trivially holds for $n = 1$, and follows from the Hurewicz theorem for $n = 2$ (see [1], p. 67).

Finally, one could ask in connection with Theorem 2:

P 391. *Does the inequality*

$$0 < \dim\{y: n - \dim f(S_n) \leqslant \dim f^{-1}(y)\}$$

hold for every non-constant mapping f *of the sphere* S_n $(n = 3, 4, \ldots)$?

Since $0 < n$ yields $0 < \dim f(S_n)$ for any non-constant f, the set $\{\ \}$ in P 391 is equal to $f(S_n)$ for $n = 1$, and to $f(S_n)$ or $\{y: 0 < \dim f^{-1}(y)\}$ for $n = 2$. Thus, for $n = 1$ or 2, we get the inequality in P 391, according to Theorem 2.

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ON THE L^p-SPACE OF A LOCALLY COMPACT GROUP

BY

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In the paper [5] Żelazko has shown that if G is a locally compact Abelian group which is Hausdorff, then $L^p(G)$ for $p > 1$ is an algebra under convolution if and only if G is compact. In this paper I extend this result to the case when G is discrete but not Abelian and $p \geqslant 2$. In the commutative case a new proof is given for the fact that $L^2(G)$ is an algebra under convolution if and only if G is compact, based on only measure theoretic considerations and Fourier transform. Theorem 1 is of its own interest and the author has not seen any published statement of it so far. I wish to express my thanks to Professor Ionescu Tulcea who drew my attention to the paper [5].

Measure theoretic notions are generally taken from [1]. Group theoretic notions are as found in either [2] or [3].

If (X, Σ, μ) is a measure space we write $L^p(X)$ or $L^p(\mu)$ for the space of complex valued functions $f(x)$ on X such that $\int_X |f(x)|^p d\mu(x) < \infty$, where $p \geqslant 1$ and $\neq \infty$. Similarly, $L^\infty(X)$ or $L^\infty(\mu)$ will denote the space of all essentially bounded measurable functions on X. If $f(x) \in L^p(X)$, then $\|f\|_p$ will denote the usual norm in $L^p(X)$ for $p \geqslant 1$. If G is a group with a left Haar measure μ, then $f * g$ will denote the convolution product $\int_G f(y^{-1}x)g(y)d\mu(y)$ provided $f(x)$ and $g(x)$ are measurable and the integral exists for almost all $x \in G$.

Let (X, Σ, μ) be a measure space. A set $S \in \Sigma$ is called an *atom* if $\mu(S) \neq 0$ and if for every $E \in \Sigma$ and $\subset S$ we have either $\mu(S) = \mu(E)$ or $\mu(E) = 0$. X or μ is said to be *purely atomic* if every set of non-zero σ-finite measure can be expressed as the union of atoms. Two sets $E, F \in \Sigma$ are called *equivalent* if $\mu(E - F) = \mu(F - E) = 0$.

Hereafter we consider a fixed measure space (X, Σ, μ) until theorem 1.

Now we state the following lemma without proof:

LEMMA 1. *If every set of non-zero measure contains an atom, then* μ *is purely atomic.*

LEMMA 2. *If there exists a sequence $E_1 \supset E_2 \supset \dots$ of sets $\in \Sigma$ and of non-zero measure such that $\lim_{n\to\infty} \mu(E_n) = 0$, then $L^1(X) \not\subset L^p(X)$ for any $p > 1$. Similarly, if there exists a sequence $E_1 \subset E_2 \subset \dots$ of sets $\in \Sigma$ and of finite measure such that $\lim_{n\to\infty} \mu(E_n) = \infty$, then $L^1(X) \not\supset L^p(X)$ for any $p > 1$.*

Proof. Without loss of generality we can assume in the first case that $0 < \mu(E_n) \leqslant (\frac{1}{2})^n$ and $\mu(E_n) \neq \mu(E_{n-1})$ for $n = 2, 3, \dots$ Then define $f(x) = 0$ outside E_2 and $f(x) = (1/n^2)\mu(E_n - E_{n+1})$ in $E_n - E_{n+1}$ for all $n = 2, 3, \dots$ Then $f(x)$ belongs to L^1 but not to L^p for any $p > 1$. Similarly we can prove the other result.

THEOREM 1. (i) *For any $p > 1$ we have $L^1(X) \subset L^p(X)$ if and only if the measure is atomic, and the set of measures of all atoms has a strictly positive lower bound whenever this set is not empty.*

(ii) *For any $p > 1$ we have $L^1(X) \supset L^p(X)$ if and only if every set E of σ-finite measure has finite measure and the set of measures of all sets of finite measure is bounded above.*

Proof. (i) Let $L^1(X) \subset L^p(X)$ for some $p > 1$. Let E be any set belonging to Σ and of non-zero measure, if one such exists. Then either E is an atom or there is a set $E_1 \in \Sigma$ such that $0 < \mu(E_1) \leqslant \frac{1}{2}\mu(E)$ and $\mu(E_1) < \infty$. Now if E_1 is not an atom, then there exists a set $E_2 \subset E_1$ such that $0 < \mu(E_2) \leqslant \frac{1}{2}\mu(E_1)$. Proceeding like this we get that either E contains an atom or that there is a sequence $E_1 \supset E_2 \supset \dots$ of sets of non-zero measure such that $\lim_{n\to\infty} \mu(E_n) = 0$. But this latter possibility is ruled out by lemma 2. Hence, by lemma 1, μ is purely atomic. Hence, by lemma 2, the set of the measures of the atoms, if any, must have a strictly positive lower bound.

We can prove (ii) similarly.

THEOREM 2. *Let G be a locally compact Hausdorff topological group. Let μ be its left Haar measure. Then $L^1(G) \subset L^p(G)$ for some $p > 1$ if and only if G is discrete.*

Proof. Since G is Hausdorff and locally compact we see that given an open set U containing more than one point we can find an open set V such that $\overline{V}$ is compact and $\overline{V} \subset U$ and $\overline{V} \neq U$. From this and lemma 2 we get that if G is not discrete, then $L^1(G) \not\subset L^p(G)$ for any $p > 1$. The converse is obvious.

As a simple consequence of Theorem 2 we obtain a case of Żelazko's theorem:

COROLLARY. *If G is a locally compact Abelian Hausdorff topological group with Haar measure μ, then $L^2(G)$ is an algebra under convolution if and only if G is compact.*

Proof. If G is compact then clearly $L^2(G)$ is an algebra under convolution. Now let us assume that G is not compact. Let $\hat{G}$ be the character group of G with its Haar measure. Then G is not discrete, whence, by theorem 2, $L^1(\hat{G}) \not\subset L^2(\hat{G})$. Hence there is a function $f(\chi)$ in $L^1(\hat{G})$ and not in $L^2(\hat{G})$. Let $\varphi(x)$ be the inverse Fourier transform of $\sqrt{|f(\chi)|}$. Then $\varphi(x) \in L^2(G)$ since $\sqrt{|f(\chi)|} \in L^2(\hat{G})$. But $\varphi * \varphi \notin L^2(G)$ for if it did then its Fourier transform which is $|f(\chi)|$ should belong to $L^2(\hat{G})$ which is not the case. Hence $L^2(G)$ is not an algebra under convolution.

THEOREM 3. *Let G be a discrete group. Let μ be its Haar measure. Then $L^p(G)$ is an algebra under convolution for $p \geqslant 2$ if and only if G is finite.*

Proof. If G is finite, then clearly $L^p(G)$ is closed under convolution for all $p \geqslant 1$. Now let $L^p(G)$ $(p \geqslant 2)$ be closed under convolution. By simple reasoning we infer that the convolution is a continuous operation. The function $e_0(x)$ which is equal to one at the identity of G and zero elsewhere is an identity of $L^p(G)$. Thus the operator norm $|||f|||_p = \sup_{\|g\|_p=1} \|f * g\|_p$ is equivalent to the norm $\|f\|_p$ and, consequently,

$$\|f\|_p \leqslant |||f|||_p \leqslant C\|f\|_p$$

for a constant C. Hence we get the inequality

$$\|f_1 * f_2\|_p \leqslant |||f_1 * f_2|||_p \leqslant |||f_1|||_p |||f_2|||_p \leqslant C^2 \|f_1\|_p \|f_2\|_p,$$

which shows that by replacing the Haar measure μ by $C^2\mu$ we can make $L^p(G)$ a Banach algebra. In the sequel we shall assume that the Haar measure μ is so chosen that $\|f\|_p$ is submultiplicative.

First consider the algebra $L^2(G)$. Defining $\tilde{f}(x) = \overline{f(x^{-1})}$, $L^2(G)$ is made an H^*-algebra with unit element [2]. Hence from [4] we infer that $L^2(G)$ is finite dimensional. Therefore, G is finite.

Now let $f(x), g(x) \in L^p(G)$ $(p > 2)$ and $h(x) \in L^q(G)$, where $1/p + 1/q = 1$. Then $\int_G f(x) g(y^{-1}x) d\mu(x) \in L^p(G)$ and

$$\Big\|\int_G f(x) g(y^{-1}x) d\mu(x)\Big\|_p \leqslant \|f\|_p \|g\|_p.$$

Hence the integral

$$\int_G h(y) d\mu(y) \Big[\int_G f(x) g(y^{-1}x) d\mu(x)\Big]$$

exists and

$$\Big|\int_G f(x) d\mu(x) \Big(\int_G g(y^{-1}x) h(y) d\mu(y)\Big)\Big| \leqslant \|b\|_p \|g\|_p \|h\|_q.$$

From this we get that, if $f(x) \in L^p(G)$ and $h(x) \in L^q(G)$, then $f * h \in L^q(G)$. In particular, taking the function $e_0(x)$ above for $h(x)$ and $f(x)$ to be any function in $L^p(G)$ we get $f(x) = f * e_0 \in L^q(G)$, whence $L^p(G) \subset L^q(G)$.

But $p > 2$, and consequently $q < 2 < p$. Hence $L^q(G) \subset L^p(G)$, and consequently $L^p(G) = L^2(G) = L^q(G)$. Therefore $L^2(G)$ is an algebra under convolution.

Hence G is finite.

Note. The author has proved after submitting this paper that for any locally compact group G the space $L^p(G)$ is closed for convolution for some $p > 2$ if G is compact.

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A NOTE ON L_p-ALGEBRAS

BY

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In [3] it was shown that if G is a locally compact Abelian group, then $L_p(G)$ for $p > 1$ is a Banach algebra under convolution if and only if G is compact. Further, Rajagopalan [2] extended this result to the case when G is discrete but not Abelian and $p \geqslant 2$. In this paper we prove this result for an arbitrary locally compact group under the assumption that $p > 2$.

Let G be a locally compact group. Its elements will be denoted by t, τ; group operation will be written multiplicatively. Unit element will be de noted by e. If A, B are subsets of G, then AB is a set of all elements of G written in the form $t \cdot \tau$, where $t \in A$, $\tau \in B$, and A^{-1} is defined as the set of all t^{-1}, such that $t \in A$. U, V will stand for compact neighbourhoods of the unit e. It is known that for every neighbourhood U, there exists a symmetric neighbourhood $V \subset U$ (i. e. such that $V = V^{-1}$) for which $V^2 \subset U$. μ will denote the left invariant Haar measure on G. We recall that if A is open and B compact, then $\mu(A) > 0$, and $\mu(B) < \infty$. Generally speaking the left invariant measure is not the right invariant one, but there exists such a continuous function $\Delta(t)$, called *modular function*, that $\mu(At) = \mu(A)\Delta(t)$ for every measurable A, and $t \in G$. We have $\Delta(t) > 0$ for every $t \in G$, $\Delta(e) = 1$, and

$$\Delta(t\tau) = \Delta(t)\Delta(\tau). \tag{1}$$

In the case when $\Delta(t) \equiv 1$ the group G is called *unimodular*. In this case we have

$$\int f(t\tau)\mu(d\tau) = \int f(\tau t)\mu(d\tau) = \int f(\tau^{-1})\mu(d\tau) = \int f(\tau)\mu(d\tau) \tag{2}$$

for every integrable function f defined on G and $t \in G$. $L_p(G)$ will denote the space of all complex functions (or more exactly of equivalence classes) such that

$$\|x\|_p = \left(\int |x(t)|^p \mu(dt)\right)^{1/p} < \infty.$$