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ON MAPPINGS THAT CHANGE DIMENSIONS OF SPHERES

BY

A. LELEK (WROCŁAW)

A mapping (i.e. a continuous function) f of the space X is said to be *strongly irreducible* provided that $f(A) = f(X)$ implies $A = X$ for every closed subset A of X (see [3], p. 162).

We denote by D_f the set of all points of X on which f is 1-1, i.e.

$$D_f = \{x\colon x \in X,\, x = f^{-1}f(x)\};$$

the mapping f is obviously strongly irreducible if the set D_f is dense in X. It is known that the inverse is also true provided that X is a compact metric space (see [3], p. 163).

Examples. Let us denote by bI^n the boundary of the n-dimensional cube I^n in the n-dimensional Euclidean space E^n.

There exists a strongly irreducible monotone mapping f of I^n $(n = 2, 3, \ldots)$ such that $f(bI^n)$ is a point and $\dim f(I^n) = 1$ (1). *Hence $f(I^n)$ is a dendrite* (see [1], p. 333, 336 and 338).

Indeed, let C be the Cantor ternary set on the segment $I = \{t\colon 0 \leqslant t \leqslant 1\}$, and let P be an arbitrary n-dimensional parallelepiped in E^n, with boundary bP and centre q. Consider a set A, consisting of points $q + c(x-q)$, where $c \in C - \{1\}$, $x \in bP$, and E^n is understood to be a vector space. Then A is a nowhere dense closed subset of $P - bP$, and each component B of $(P-bP)-A$ is a domain in E^n, bounded by surfaces $q + c_i(bP-q)$, clearly homeomorphical to bP, where $i = 1, 2$, and c_1, c_2 are end points of a component interval of $I - C$. Let us cut every domain B with compact pieces of $(n-1)$-dimensional hyperplanes contained in the closure of B into a finite number of parallelepipeds P' whose diameters $\delta(P')$ are less than $\frac{1}{2}\delta(P)$. Denote by A' the union of A and of all these $(n-1)$-dimensional pieces, where B ranges over the countable collection of all components of $(P-bP)-A$. So A' is also closed in $P - bP$. Hence the collection $\boldsymbol{C}(P)$ of components of A' is one of continua

(1) The idea of the example is due to K. Sieklucki.

and remains an upper semicontinuous collection after adding to it a new element being an arbitrary continuum contained in $(E^n-P)\cup bP$ and containing bP. Moreover, E^n-A' is a $\boldsymbol{G}_\delta$-set dense in E^n, and the countable collection $\boldsymbol{P}(P)$ of closures of components of $(P-bP)-A'$ is one of parallelepipeds P' such that $\delta(P')\leqslant\frac{1}{2}\delta(P)$, and the boundary bP' is contained in some element of $\boldsymbol{C}(P)$ for every $P'\epsilon\boldsymbol{P}(P)$.

Now we define the sequence $\boldsymbol{P}_0,\boldsymbol{P}_1,\ldots$ of countable collections of parallelepipeds as follows: $\boldsymbol{P}_0=\{I^n\}$ and $\boldsymbol{P}_{k+1}$ is the union of all collections $\boldsymbol{P}(P)$, where $P\epsilon\boldsymbol{P}_k$, for $k=0,1,\ldots$ Let us observe that if $P\epsilon\boldsymbol{P}_k$, then $\delta(P)\leqslant\sqrt{n}/2^k$, as the diameter of the cube I^n is $\sqrt{n}$. Thus

$$\text{(i)}\qquad \boldsymbol{K}=\{bI^n\}\cup\bigcup_{k=0}^{\infty}\bigcup_{P\epsilon\boldsymbol{P}_k}\boldsymbol{C}(P)$$

is a collection of disjoint continua in I^n (for $n=2,3,\ldots$), and for any sequence $K_1,K_2,\ldots$ of elements of $\boldsymbol{K}$ such that $K_i\epsilon\boldsymbol{C}(P_i)$, where $P_i\epsilon\boldsymbol{P}_{k_i}$ and $k_i\to\infty$, we have $\delta(K_i)\to 0$. It follows that the collection $\boldsymbol{L}$ consisting of all elements of $\boldsymbol{K}$ and of single points of $I^n-\boldsymbol{K}^*$ (2) is upper semicontinuous and $I^n=\boldsymbol{L}^*$.

Let f be the mapping induced by $\boldsymbol{L}$, i. e. satisfying $f^{-1}(y)\epsilon\boldsymbol{L}$ for every $y\epsilon f(I^n)$ (see [1], p. 42). Then f is monotone. Further, we have $D_f=I^n-\boldsymbol{K}^*$, all sets $I^n-\boldsymbol{C}^*(P)$ are $\boldsymbol{G}_\delta$-sets dense in I^n for $P\epsilon\boldsymbol{P}_k$, $k=0,1,\ldots,$ and their intersection is $(I^n-\boldsymbol{K}^*)\cup bI^n$, according to (i). Hence D_f is dense in I^n by the Baire theorem, and so f is strongly irreducible. As $bI^n\epsilon\boldsymbol{K}\subset\boldsymbol{L}$, $f(bI^n)$ is a point. Each element L of $\boldsymbol{L}$ can be separated from any closed subset of I^n which does not meet L, with a finite number of elements of $\boldsymbol{L}$. Thus the continuum $f(I^n)$ has finite order of ramification at any point, and therefore it is 1-dimensional.

Using the above example one can easily show that *if P is a polyhedron with $2\leqslant\dim_pP$ for every $p\epsilon P$, then there exists a strongly irreducible monotone mapping f of P such that* $\dim f(P)=1$.

In particular, we get such a mapping f for P being the n-dimensional sphere S_n with $n=2,3,\ldots$ (3).

Theorems. It can be verified that in the last example the points of S_n on which f is 1-1 correspond to the end points of the dendrite $f(S_n)$, i.e. $f(D_f)$ is the end point set of $f(S_n)$, dense in $f(S_n)$. However, for mappings f which lower the dimension of S_n, the set $f(D_f)$ cannot be too large.

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(2) If $\boldsymbol{K}$ is a collection of sets, $\boldsymbol{K}^*$ denotes the union of all elements belonging to $\boldsymbol{K}$.

(3) This answers a question proposed to the author by L. V. Keldysh in September 1961 during the International Topological Symposium in Prague.

THEOREM 1. *If f is a non-constant mapping of the sphere S_n $(n=0,1,\ldots)$ and* $\dim f(S_n-D_f)\leqslant 0$, *then* $n\leqslant\dim f(S_n)$.

Proof. The theorem being evidently true for $n=0$, let us assume that $n>0$. Since f is a non-constant mapping, $f(S_n)$ is a non-degenerate continuum. Thus, if the set D_f were finite, the set $f(S_n-D_f)=f(S_n)-f(D_f)$ would have positive dimension, contrary to the hypothesis. Hence D_f is infinite.

Let $p,q\epsilon D_f$ and $p\neq q$. The further proof is the same as in [2], pages 79-81, for $X=D_f$, $Y=f(S_n)$ and $h=f|D_f$. Beginning at page 81, line 14 from bottom, where the hypothesis is used that X is densely connected in S_n (which is not assumed here), the proof must be modified by replacing $h(R\cap X)$ by $f(R)$. Then, though the set $R\cap X$ needs not be connected and therefore inclusion (17) is useless, the proof remains valid. In fact, the set R being connected and containing the points p and q, we infer in the same way that $C_{i_0}\cap f(R)\neq 0$ for some $i_0=1,\ldots,j$. Hence $f(B)\cap f(S_n-B)\neq 0$, because $C_{i_0}\subset h(B)=f(B)$ and $R\subset S_n-B$. However, the mapping f is 1-1 on B as $B\subset X=D_f$, and so $f(B)\cap f(S_n-B)=0$, which yields the desired contradiction.

THEOREM 2. *If f is a non-constant mapping of the sphere S_n $(n=1,2,\ldots)$ and* $\dim f(S_n)<n$, *then*

$$0<\dim\{y\colon 0<\dim f^{-1}(y)\}.$$

Proof. Denote by Y the set $\{\ \}$ in the last inequality. Let $f=gh$ be the decomposition of f into mappings g and h such that h is monotone and g is 0-dimensional, i.e. $\dim g^{-1}(y)=0$ for every $y\epsilon f(S_n)$ (see [1], p. 125). Then if $x\epsilon h(S_n-D_h)$, the set $h^{-1}(x)$ is non-degenerate, whence $0<\dim h^{-1}(x)$ by the monotoneity of h. But since $x\epsilon g^{-1}g(x)$, we obtain

$$h^{-1}(x)\subset h^{-1}g^{-1}g(x)=f^{-1}g(x),$$

which gives $0<\dim f^{-1}g(x)$, that is $g(x)\epsilon Y$. Thus

$$\text{(ii)}\qquad gh(S_n-D_h)\subset Y.$$

Since the mapping g is 0-dimensional, we have

$$\dim h(S_n)\leqslant\dim gh(S_n)=\dim f(S_n)<n,$$

according to the Hurewicz theorem (see [1], p. 67). This implies

$$\text{(iii)}\qquad 0<\dim h(S_n-D_h),$$

by virtue of Theorem 1. But, D_h being a $\boldsymbol{G}_\delta$-set (see [3], p. 162), its complementary set, as well as the continuous image $h(S_n-D_h)$, is a $\boldsymbol{F}_\sigma$-set, i.e. the union of countably many compact sets. It follows from (iii) that there exists a compact set $X \subset h(S_n - D_h)$ with $0 < \dim X$. So $\dim X \leqslant \dim g(X)$ by the same Hurewicz theorem, and $g(X) \subset Y$ by (ii). We get $0 < \dim Y$.

Questions. We have shown by the example (see p. 46) that for some mapping f which lowers the dimension of S_n the set D_f can be dense in S_n. Then $\dim(S_n - D_f) \leqslant n-1$ (see [1], p. 353). Actually, the set $S_n - D_f$ has the dimension equal to $n-1$. This suggests the following question:

P 390. *Is it true that* $\dim f(S_n) \leqslant n-1$ *implies* $n-1 \leqslant \dim(S_n - D_f)$ *for every mapping* f *of the sphere* S_n $(n = 3, 4, \ldots)$?

The proposition trivially holds for $n = 1$, and follows from the Hurewicz theorem for $n = 2$ (see [1], p. 67).

Finally, one could ask in connection with Theorem 2:

P 391. *Does the inequality*

$$0 < \dim\{y: n - \dim f(S_n) \leqslant \dim f^{-1}(y)\}$$

hold for every non-constant mapping f *of the sphere* S_n $(n = 3, 4, \ldots)$?

Since $0 < n$ yields $0 < \dim f(S_n)$ for any non-constant f, the set $\{\ \}$ in P 391 is equal to $f(S_n)$ for $n = 1$, and to $f(S_n)$ or $\{y: 0 < \dim f^{-1}(y)\}$ for $n = 2$. Thus, for $n = 1$ or 2, we get the inequality in P 391, according to Theorem 2.

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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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ON THE L^p-SPACE OF A LOCALLY COMPACT GROUP

BY

M. RAJAGOPALAN (NEW HAVEN, CONN.)

In the paper [5] Żelazko has shown that if G is a locally compact Abelian group which is Hausdorff, then $L^p(G)$ for $p > 1$ is an algebra under convolution if and only if G is compact. In this paper I extend this result to the case when G is discrete but not Abelian and $p \geqslant 2$. In the commutative case a new proof is given for the fact that $L^2(G)$ is an algebra under convolution if and only if G is compact, based on only measure theoretic considerations and Fourier transform. Theorem 1 is of its own interest and the author has not seen any published statement of it so far. I wish to express my thanks to Professor Ionescu Tulcea who drew my attention to the paper [5].

Measure theoretic notions are generally taken from [1]. Group theoretic notions are as found in either [2] or [3].

If (X, Σ, μ) is a measure space we write $L^p(X)$ or $L^p(\mu)$ for the space of complex valued functions $f(x)$ on X such that $\int_X |f(x)|^p d\mu(x) < \infty$, where $p \geqslant 1$ and $\neq \infty$. Similarly, $L^\infty(X)$ or $L^\infty(\mu)$ will denote the space of all essentially bounded measurable functions on X. If $f(x) \in L^p(X)$, then $\|f\|_p$ will denote the usual norm in $L^p(X)$ for $p \geqslant 1$. If G is a group with a left Haar measure μ, then $f * g$ will denote the convolution product $\int_G f(y^{-1}x) g(y) d\mu(y)$ provided $f(x)$ and $g(x)$ are measurable and the integral exists for almost all $x \in G$.

Let (X, Σ, μ) be a measure space. A set $S \in \Sigma$ is called an *atom* if $\mu(S) \neq 0$ and if for every $E \in \Sigma$ and $\subset S$ we have either $\mu(S) = \mu(E)$ or $\mu(E) = 0$. X or μ is said to be *purely atomic* if every set of non-zero σ-finite measure can be expressed as the union of atoms. Two sets $E, F \in \Sigma$ are called *equivalent* if $\mu(E-F) = \mu(F-E) = 0$.

Hereafter we consider a fixed measure space (X, Σ, μ) until theorem 1.

Now we state the following lemma without proof:

LEMMA 1. *If every set of non-zero measure contains an atom, then* μ *is purely atomic.*