MAPPINGS OF INVERSE LIMITS

by

J. MIODUSZEWSKI (WROCŁAW)

The purpose of this note is to give necessary and sufficient conditions for a compact metric space to be a continuous image of another one expressed in terms of inverse expansions in polyhedra. Also conditions for homeomorphism are given. These are analogous to the conditions given by Alexandrov [1] and Švedov [5] for another kind of inverse expansions. The results of this note have applications in [4].

1. Preliminaries. We consider inverse limits in the sense of [2]. Let \( X = \lim_{\to} (X_n, \pi_n^m, M) \), where \( n, m \in M \), \( M \) is a directed set, \( \pi_n^m: X_n \to X_m, n \geq m \), are continuous mappings (¹) and \( X_n \) are polyhedra. We denote by \( \pi_n \) projections from \( X \) into \( X_n \). Let \( Y = \lim_{\to} (Y_n, \sigma_n^m, N) \) be another such a limit. Let \( f: X \to Y \) be a mapping. We shall use the following approximation lemma of [2] (Theorem XI. 12. 9 with Lemmas XI. 3.1 and X. 3.8) which may be expressed, for the case considered here, as follows:

**Lemma 1.** For every \( n \in N \) and \( \varepsilon > 0 \) there exists \( m_0 \in M \) such that for every \( m \geq m_0 \) there exists a mapping \( f_{mn}: X_m \to Y_n \) such that the diagram

\[
\begin{array}{ccc}
X_m & \to & X \\
\downarrow & & \downarrow \\
Y_n & \to & Y
\end{array}
\]

is \( \varepsilon \)-commutative, i.e. the distance between \( f_{mn} \pi_n(X) \) and \( \sigma_n f(x) \) is less than \( \varepsilon \) for every \( x \in X \).

2. Mappings of compact metric spaces. We consider now inverse systems of polyhedra \( (X_n, \pi_n^m) \), where \( \pi_n^m \) are onto and \( m, n \) are positive integers. According to Preudenthal [3], every compact metric space is an inverse limit of such a system.

**Theorem 1.** If a space \( Y = \lim X_n, \sigma_n^m \) is a continuous image of

(¹) Throughout this note all mappings are assumed to be continuous.
a space $X = \lim_{n \to \infty} (X_n, \pi^n)$, then for every sequence $\{e_n\}$, where $e_n > 0$ and $\lim e_n = 0$, there exists an infinite diagram

\[
\begin{align*}
X_{m_k} \leftarrow X_{m_{k-1}} \leftarrow \cdots \leftarrow X_{m_2} \leftarrow X_{m_1} \leftarrow \cdots \\
Y_{m_1} \leftarrow Y_{m_2} \leftarrow \cdots \leftarrow Y_{m_3} \leftarrow Y_{m_4} \leftarrow \cdots
\end{align*}
\]

(2)

where $\{m_k\}$ and $\{n_k\}$ are non-decreasing and unbounded sequences of positive integers, and every subdiagram of the form

\[
\begin{align*}
X_{n_k} \leftarrow X_{n_{k-1}} \\
Y_{n_k} \leftarrow Y_{n_{k-1}}
\end{align*}
\]

(3)

is $\epsilon_k$-commutative for all $k \leq k$ and $r \geq k$.

**Proof.** Let $Y = f(X)$. We define the required diagram by induction. Let $n_1 = 1$. According to Lemma 1, we choose $m_1$ and $f_1 : X_{m_1} \to Y_{m_1}$ such that diagram (1) is $\epsilon_1$-commutative for $m = m_1$, $n = n_1$ and $f_{m_1} = f_1$.

Suppose that $m_k$, $n_k$ and $f_k$ are already defined for $k \leq j$ and that they have the following properties:

1° subdiagrams (3) lying in the already constructed part of (2) are $\epsilon_k$-commutative,

2° diagrams (1) for $n = n_k$, $m = m_k$ and $f_{m_1} = f_k$ are $\epsilon_k$-commutative.

Let $m_{k+1} \geq n_k$. Note first that there exists $r_k > 0$ such that $\eta$-commutativity of diagrams (1) for $\eta \leq r_k$, $n = n_{k+1}$ and $m \geq m_k$ implies $\epsilon_k$-commutativity of diagrams (3) with $m = m_k$, $n = n_{k+1}$ and $k \leq j$. Choose $\eta = \min(\eta, r_k)$, and then $m_{k+1} \geq m_k$ and $f_{k+1} : X_{m_{k+1}} \to Y_{n_k}$ are such that the diagram (1) is $\eta$-commutative for $n = n_{k+1}$, $m = m_{k+1}$ and $f_{m_1} = f_{k+1}$. It is easy to verify that the properties 1° and 2° hold for $k \leq j+1$.

**Remark.** If $Y$ is of dimension 0, then diagrams of Theorem 1 may be taken simply commutative. It would be interesting to know whether it is possible to obtain the commutativity in Theorem 1 for the dimension of $Y$ greater than 0 (P. 399).

A special case of the following theorem is known from [2] (Theorem VIII.3.11). It gives a sufficient condition for a compact metric space to be a continuous image of another one. We shall write $f \psi g$ if the distance between $f(x)$ and $g(x)$ is less than $\varepsilon$ for all $x$.

**Theorem 2.** Let $\{e_n\}$, $n = 1, 2, \ldots$, be a sequence of positive numbers such that $\lim e_n = 0$. The existence of an infinite diagram (2) having, with respect to this sequence, the properties required in Theorem 1 induces the existence of continuous mapping $f : X \to Y$ such that $\epsilon_n f_{m_n} \psi \epsilon_n f$ for every $n$ and $x$, $x \leq m_n$, and which is onto if $f_k$ are onto.

**Proof.** We define $f$ as a mapping which sends $x = (x_1, x_2, \ldots)$ onto $y = (y_1, y_2, \ldots)$, where $y_n = \lim_{k \to \infty} \epsilon_n f_k(x_{m_k})$. The limit exists because for $r \geq k$ we have $\epsilon_n f_k(x_{m_k}) \to \epsilon_n f_k(x_{m_k})$ according to $\epsilon_k$-commutativity of diagram (3). The point $y$ defined in this way belongs to $Y$, as

\[
\begin{align*}
c_1(y_1) &= \epsilon_1(\lim_{k \to \infty} \epsilon_n f_k(x_{m_k})) = \lim_{k \to \infty} \epsilon_n f_k(x_{m_k}) \\
&= \lim_{k \to \infty} \epsilon_n f_k(x_{m_k}) = y_1, \quad i = 1, 2, \ldots
\end{align*}
\]

The $\epsilon_i$-equalities required by the Theorem are valid according to the definition of $f$.

In order to prove the continuity of $f$, it is sufficient to prove the continuity of $\epsilon f$ for every $s = 1, 2, \ldots$. Let $s > 0$ and $\varepsilon$ be given. Choose $k$ such that $\epsilon_n f_k(x_{m_k}) \psi (\varepsilon/s)$. Let $x'$ and $x''$ be such that the distance between $\epsilon_n f_k(x''_{m_k})$ and $\epsilon_n f_k(x'_{m_k})$ is not greater than $\varepsilon$. Then the distance between $\epsilon_n f(x'(x'))$ and $\epsilon_n f(x''(x''))$ is not greater than $3s$. The continuity is proved.

Now assume that $f_k$ are onto. Let $y = (y_1, y_2, \ldots) \in Y$. As $f_k$ are onto, then $\epsilon_n f_k(x_{m_k})$ is a non-empty set for every $s$ and $x \leq m_n$. We denote by $A_s$ the topological limit superior of these sets if $k \to \infty$. It is easy to verify that if $x \in A_s$, then $f(x) = y_s$. Note that $A_s \subseteq A_r$, for $s \geq r$.

Let then $x \in \bigcap_{s=1}^{m_n} A_s$. Then $\epsilon_n f(x) = y$, for every $s$, i.e. $f(x) = y$.

More convenient for applications is the following weaker form of the above Theorem. Let $\mathcal{F}$ be a class of mappings.

**Theorem 2'.** If for every pair of positive integers $m$ and $n$, for every mapping $f_{m_1} : X_m \to Y_m$ belonging to $\mathcal{F}$, and for every $\varepsilon > 0$ and $n > m$ there exist $m' > m$ and a mapping $f_{m' \to m} : X_{m'} \to X_m$ belonging to $\mathcal{F}$ such that the diagram

\[
\begin{align*}
X_m \leftarrow X_{m'} \\
Y_m \leftarrow Y_{m'}
\end{align*}
\]

(4)

is $\varepsilon$-commutative, then there exists a mapping $f : X \to Y$ which is onto if all mappings in $\mathcal{F}$ are onto.

The proof reduces to the verification that the hypotheses of Theorem 2' implies the hypotheses of Theorem 2, i.e. to the construction of diagram (9). This construction is made by induction which is standard and therefore will be omitted.
3. Homeomorphism of compact metric spaces. We prove now

**Theorem 3.** If \( X = \lim (X_n, x_n^n) \) and \( Y = \lim (Y_n, y_n^n) \) are homeomorphic, then for every sequence \( (\varepsilon_n) \) such that \( \varepsilon_n > 0 \) and \( \lim \varepsilon_n = 0 \), there exists an infinite diagram

\[
\begin{align*}
X_{n_1} & \leftarrow X_{n_2} \leftarrow \cdots \leftarrow X_{n_{m_1}} \leftarrow \cdots \\
Y_{n_1} & \leftarrow Y_{n_2} \leftarrow \cdots \leftarrow Y_{n_{m_1}} \leftarrow \cdots,
\end{align*}
\]

(5)

where \( (n_k) \) and \( (m_k) \) are unbounded and non-decreasing sequences of positive integers, and every subdiagram of the form

\[
\begin{align*}
X_{n_k} & \leftarrow X_{n_k} \leftarrow \cdots \leftarrow X_{n_{m_k}} \leftarrow \cdots \\
Y_{n_k} & \leftarrow Y_{n_k} \leftarrow \cdots \leftarrow Y_{n_{m_k}} \leftarrow \cdots,
\end{align*}
\]

(5’)

is \( \varepsilon_k \)-commutative in the cases (5’ and 5’’) and \( \varepsilon_k \)-commutative in the cases (5’’ and 5’’’).

**Proof.** Let \( Y = f(X) \) and \( X = g(Y) \), where \( f \) and \( g \) are identities. We construct the required diagram by induction. Let \( n_1 = 1 \). According to Lemma 1, we choose \( m_1 \) and \( f_1: X_{n_1} \rightarrow Y_{n_1} \) such that diagram (1) is \( \varepsilon_1 \)-commutative for \( m = m_1, n = n_1 \) and \( f = f_1 \). Suppose that \( m_{k-1}, n_{k-1}, m_k, n_k, f_k, g_k \) are already defined for indices \( 2k-1 \) and \( 2k \) not greater than \( j = 2p-1 \) (the case \( j = 2p \) is symmetric to this one) and that they have the following properties:

1. Subdiagrams (5’’), (5’’’) lying in the already constructed part of (5) are \( \varepsilon_{k-1} \)-commutative and \( \varepsilon_k \)-commutative respectively,

2. Diagonals (2) for \( f_{nm} = f_k \) are \( \varepsilon_k \)-commutative and similar diagrams for \( g_k \) and \( g \) are \( \varepsilon_k \)-commutative.

Let \( m_{2p} \geq m_{2p-1} \). Note that there exist \( \eta_p > 0 \) such that \( \eta \)-commutativity of the diagram of type (1) for \( \eta \leq \eta_p \), for mapping \( g \) instead of \( f \) and for mapping \( g_{2p} \) instead of \( f_{2p} \), where \( m_{2p-1} \geq \eta_p \) implies the \( \varepsilon_k \)-commutativity and \( \varepsilon_{k-1} \)-commutativity of diagrams (5’), (5’’), (5’’’), and \( \varepsilon_{k-1} \)-commutativity of diagrams (5’), (5’’), (5’’’), and (5’’’) for \( \eta = \eta_p \), \( m = m_2 \), and all \( k \) such that \( 2k \) and \( 2k - 1 \) is not greater than \( j = 2p-1 \). We choose \( \eta \leq \min (\eta_1, \eta_2) \) and then we choose \( m_{2p} \geq m_{2p-1} \) and \( g_k: X_{2p} \rightarrow X_{2p} \) such that the diagram of type (1) for \( g_k \) and \( \eta \) is \( \eta \)-commutative. It is easy to verify that the properties 1’ and 2’ hold for \( 2k \) and \( 2k - 1 \) not greater than \( 2p \).

**Remark.** As in the case of Theorem 1, diagram (4) may be taken simply commutative if \( X \) and \( Y \) are of dimension 0.

**Theorem 4.** Let \( (\varepsilon_n) \) be a sequence of positive numbers such that \( \lim \varepsilon_n = 0 \). The existence of an infinite diagram (4) having, with respect to this sequence, the properties required in Theorem 3 induces the existence of a homeomorphism \( f \) of \( X \) onto \( Y \) (the inverse of \( f \) is denoted by \( g \)) such that \( \varepsilon_{k-1} \cdot f_k. X_{2k-1} \rightarrow Y_{2k-1} \) for every \( k \) and \( \varepsilon_k \in \varepsilon_{k-1} \) in the first case, and \( k, \varepsilon_k \leq m_{2k-1} \) in the second one.

According to Theorem 2, diagram (4) induces the existence of mappings \( f: X \rightarrow Y \) and \( g: Y \rightarrow X \). It remains to show that \( fg \) and \( gf \) are identities. We shall verify only the first inequality. According to the definition of \( f \) and \( g \) (see the proof of Theorem 2) we have

\[
\begin{align*}
\sigma_k fg(y) &= \lim_{b \to \infty} \sigma_{k-1} f_k \lim_{n \to \infty} g_{n_k}(x_{n_k}) \\
&= \lim_{b \to \infty} \sigma_{k-1} f_k \lim_{n \to \infty} g_{n_k}(y_{n_k}) \\
&= \lim_{b \to \infty} \sigma_{k-1} (y_{n_k})
\end{align*}
\]

where \( y_{n_k} \rightarrow y_{n_k} \) by \( \varepsilon_k \)-commutativity of diagram (5’). By commutativity \( (\varepsilon_k) \)-commutativity of \( g_{n_k} \) also \( \varepsilon \)-commutativity of \( \sigma_{k-1} \). Hence, we have \( \lim_{b \to \infty} \sigma_{k-1} (y_{n_k}) = y_k \). Thus, \( fg(y) = y \) for every \( y \), i.e., \( fg \) is the identity.

The more convenient for applications is the following weaker form of the above Theorem. Let \( \Sigma \) and \( \eta \) be classes of mappings.

**Theorem 4’.** If for every pair of positive integers \( m \) and \( n \), for every mapping \( f: X \rightarrow Y \) belonging to \( \Sigma \), for every \( e > 0 \) and \( \lambda \in (0, e) \), there exists \( \lambda \rightarrow 0 \) and a mapping \( g_{nm}: Y \rightarrow X \) belonging to \( \eta \) such that the diagram

\[
\begin{align*}
X_n & \rightarrow X_m \\
Y_n & \rightarrow Y_m
\end{align*}
\]

is \( \varepsilon \)-commutative, and the same is true after change \( X \) into \( Y \), \( \Sigma \) into \( \eta \) etc., then there exists a homeomorphism between \( X \) and \( Y \).

The proof reduces to the verification that the hypotheses of Theorem 4’ implies hypotheses of Theorem 4, i.e., to the construction of diagram (4). The construction is made by induction which is standard and therefore will be omitted.

**References**


ON MAPPINGS THAT CHANGE DIMENSIONS OF SPHERES

BY

A. LELEK (WROCLAW)

A mapping (i.e., a continuous function) \( f \) of the space \( X \) is said to be strongly irreducible provided that \( f(A) = f(X) \) implies \( A = X \) for every closed subset \( A \) of \( X \) (see [3], p. 162).

We denote by \( D_f \) the set of all points of \( X \) on which \( f \) is 1-1, i.e.,

\[
D_f = \{ x : x \in X, \; x = f^{-1}(f(x)) \}
\]

the mapping \( f \) is obviously strongly irreducible if the set \( D_f \) is dense in \( X \).

It is known that the inverse is also true provided that \( X \) is a compact metric space (see [3], p. 163).

Examples. Let us denote by \( bI^n \) the boundary of the \( n \)-dimensional cube \( I^n \) in the \( n \)-dimensional Euclidean space \( E^n \).

There exists a strongly irreducible monotone mapping \( f \) of \( I^n \) (\( n = 2, 3, \ldots \)) such that \( f(bI^n) \) is a point and \( \dim f(I^n) = 1 \) (1). Hence \( f(I^n) \) is a dendrite (see [1], p. 333, 336 and 338).

Indeed, let \( C \) be the Cantor ternary set on the segment \( I = \{ t : 0 \leq t \leq 1 \} \), and let \( F \) be an arbitrary \( n \)-dimensional parallelepiped in \( E^n \), with boundary \( bF \) and centre \( q \). Consider a set \( A \), consisting of points \( q + c(x - q) \), where \( c \in C - \{ 0 \} \), \( x \in bF \), and \( b^n \) is understood to be a vector space. Then \( A \) is a nowhere dense closed subset of \( P - bF \), and each component \( B \) of \( (P - bF) - A \) is a domain in \( E^n \), bounded by surfaces \( q + c_i \), \( (bF - q) \), clearly homeomorphic to \( bF \), where \( i = 1, 2 \), and \( c_1, c_2 \) are end points of a component interval of \( I - C \). Let us cut every domain \( B \) with compact pieces of \( (n - 1) \)-dimensional hyperplanes contained in the closure of \( B \) into a finite number of parallelepipeds \( P' \) whose diameters \( \delta(P') \) are less than \( \frac{1}{4} \delta(P) \). Denote by \( A' \) the union of \( A \) and of all these \( (n - 1) \)-dimensional pieces, where \( B \) ranges over the countable collection of all components of \( (P - bF) - A \). So \( A' \) is also closed in \( P - bF \). Hence the collection \( C(P) \) of components of \( A' \) is one of continua.

(1) The idea of the example is due to K. Sieklucki.