COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 2

ON A CERTAIN PROPERTY OF DETERMINANT SYSTEMS

BY

A. BURACZEWSKI (WARSAW)

1. Introduction. R. Sikorski [2] has given some formulas for solutions of a linear equation

$$(1) \qquad (I+T)x = x_0$$

in a linear space X, and the adjoint equation

$$\xi(I+T) = \xi_0$$

in a conjugate space Ξ.

However, these formulas do not coincide with the formulas for solutions of classical Fredholm integral equation in the case where X is the space of all continuous functions, and T is an integral operator.

Such formulas have been investigated by Grothendieck [1] in the case where X is the Banach space and T is nuclear. More general formulas of this type have been given by R. Sikorski [3] in the case where X is the Banach space and T is a quasi-nuclear operator.

The purpose of this paper is to give a certain property of determinant systems for Fredholm operators of the type $I+T$ in a linear space. This property enables us to obtain formulas for solutions of the equations (1) and (1′) which are abstract analogues of the original Fredholm formulas.

The possibility of proving this property by an algebraic argument was suggested at the seminar by Prof. R. Sikorski, to whom the author is very much indebted.

2. Terminology and notation. Let Ξ and X be two linear spaces over the real or complex field $\mathfrak{F}$. The letters ξ, η, ζ will always denote elements of Ξ and the letters x, y, z — elements of X. Every mapping into $\mathfrak{F}$ will be called *functional*.

Following R. Sikorski [2] we suppose that Ξ and X are conjugate, i. e. there exists a bilinear functional on $\Xi \times X$ whose value at a point (ξ, x) is denoted by ξx and which satisfies two conditions:

(a) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$;

(a′) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

Let A be a bilinear functional on $\Xi \times X$. The value of A at a point (ξ, x) will be denoted by $\xi A x$.

In the sequel $\mathfrak{A}$ will denote the class of all bilinear functionals on $\Xi \times X$ such that the following two conditions are satisfied:

(b) for every fixed $x \in X$, there exists $y \in X$ such that $\xi A x = \xi y$ for every $\xi \in \Xi$;

(b′) for every fixed $x \in X$ there exists an $\eta \in \Xi$ such that $\xi A x = \eta x$ for every $x \in X$.

It follows from (a) that for a given x there exists exactly one y satisfying (b) and denoted by Ax. Thus every $A \in \mathfrak{A}$ determines an endomorphism $y = Ax$.

Similarly, it follows from (a′) that for a given ξ there exists exactly one η satisfying (b′) and denoted by ξA. Thus every $A \in \mathfrak{A}$ determines an endomorphism $\eta = \xi A$. By definition

$$\xi(Ax) = \xi A x \quad \text{for every } \xi \in \Xi,$$

$$(\xi A)x = \xi A x \quad \text{for every } x \in X.$$

Thus, every bilinear functional $A \in \mathfrak{A}$ can be simultaneously interpreted as endomorphism in X or as endomorphism in Ξ and conversely. Let $\xi I x = \xi x$ for every $(\xi, x) \in \Xi \times X$. Then $I \in \mathfrak{A}$ and $Ix = x$ for every $x \in X$, and $\xi I = \xi$ for every $\xi \in \Xi$.

If $A_1, A_2 \in \mathfrak{A}$, then

$$\xi A_1(A_2 x) = \xi A_1(A_2)x$$

is a bilinear functional in $\mathfrak{A}$ which will be denoted by $A_1 A_2$. Obviously, bilinear functional $A_1 A_2$ obtained in such a way can be interpreted as a superposition of endomorphisms A_1, A_2 in X or in Ξ.

Clearly, the class $\mathfrak{A}$ with the product $A_1 A_2$ defined above $(A_1, A_2 \in \mathfrak{A})$ is a linear ring.

Let ξ_0 and x_0 be fixed. The bilinear functional K_0 defined by the formula

$$\xi K_0 x = \xi x_0 \cdot \xi_0 x \ (^1)$$

will be called *one-dimensional* and will be denoted by $x_0 \cdot \xi_0$. By definition

$$K_0 x = x_0 \cdot \xi_0 x \quad \text{and} \quad \xi K_0 = \xi x_0 \cdot \xi_0.$$

Any finite sum $K = \sum_{i=1}^{n} x_i \cdot \xi_i$ of one-dimensional bilinear functionals will be called *finitely dimensional*.

(1) $\xi x_0 \cdot \xi_0 x$ means the product of numbers ξx_0 and $\xi_0 x$.

Let us consider $T \in \mathfrak{A}$ such that $I+T$ is a Fredholm bilinear functional (2) (endomorphism) of order r. Let $\zeta_1, \ldots, \zeta_r$ and $z_1, \ldots, z_r$ be linearly independent solutions of the equations $\xi(I+T) = 0$, $(I+T)x = 0$, respectively, and let $B \in \mathfrak{A}$ be a quasi-inverse (3) of $I+T$, i. e.

$$(2) \qquad (I+T)B(I+T) = I+T, \qquad B(I+T)B = B.$$

It follows from (2) that for a fixed B and fixed $\zeta_1, \ldots, \zeta_r$ and $z_1, \ldots, z_r$, there exist uniquely determined points $y_1, \ldots, y_r$, and $\eta_1, \ldots, \eta_r$, satisfying the condition

$$(3) \qquad \zeta_i y_j = \delta_{i,j}, \quad \eta_i z_j = \delta_{i,j} \quad (i, j = 1, \ldots, r),$$

and such that the following identities hold:

$$(4) \qquad (I+T)B = I - \sum_{i=1}^{r} y_i \cdot \zeta_i, \qquad B(I+T) = I - \sum_{i=1}^{r} z_i \cdot \eta_i.$$

Having (4) we easily obtain the formula

$$(5) \qquad TB - \sum_{i=1}^{r} z_i \cdot \eta_i = BT - \sum_{i=1}^{r} y_i \cdot \zeta_i.$$

The bilinear functional $I+T$, as a Fredholm one, has a determinant system (4) $D_0, D_1, \ldots$ Since the determinant system for $I+T$ is determined by T up to a scalar factor $\neq 0$, it is sufficient to suppose it to be of the form (see R. Sikorski [2])

$$(6) \qquad D_n = 0 \quad \text{for} \quad n = 0, \ldots, r-1,$$

$$(7) \qquad D_r\begin{pmatrix} \xi_1 & \ldots & \xi_r \\ x_1 & \ldots & x_r \end{pmatrix} = \begin{vmatrix} \xi_1 z_1 & \ldots & \xi_1 z_r \\ \ldots & \ldots & \ldots \\ \xi_r z_1 & \ldots & \xi_r z_r \end{vmatrix} \cdot \begin{vmatrix} \zeta_1 x_1 & \ldots & \zeta_1 x_r \\ \ldots & \ldots & \ldots \\ \zeta_r x_1 & \ldots & \zeta_r x_r \end{vmatrix},$$

and for $k = 1, 2, \ldots$

$$(8) \qquad D_{r+k}\begin{pmatrix} \xi_1 & \ldots & \xi_{r+k} \\ x_1 & \ldots & x_{r+k} \end{pmatrix} = \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn} \mathfrak{p} \operatorname{sgn} \mathfrak{q} \begin{vmatrix} \xi_{p_1} B x_{q_1} & \ldots & \xi_{p_1} B x_{q_k} \\ \ldots & \ldots & \ldots \\ \xi_{p_k} B x_{q_1} & \ldots & \xi_{p_k} B x_{q_k} \end{vmatrix} \cdot D_r\begin{pmatrix} \xi_{p_{k+1}} & \ldots & \xi_{p_{k+r}} \\ x_{q_{k+1}} & \ldots & x_{q_{k+r}} \end{pmatrix},$$

(2) For this notion see R. Sikorski [2].

(3) Every Fredholm bilinear functional has a quasi-inverse, see R. Sikorski [2].

(4) For this notion see R. Sikorski [2].

where $\sum\limits_{\mathfrak{p},\mathfrak{q}}$ is extended over all the permutations $\mathfrak{p} = (p_1, \ldots, p_{r+k})$ and $\mathfrak{q} = (q_1, \ldots, q_{r+k})$ of the integers $1, \ldots, r+k$ such that

$$\begin{aligned} &p_1 < p_2 < \ldots < p_k, \quad p_{k+1} < p_{k+2} < \ldots < p_{k+r}, \\ &q_1 < q_2 < \ldots < q_k, \quad q_{k+1} < q_{k+2} < \ldots < q_{k+r}. \end{aligned} \tag{9}$$

3. The fundamental property of determinant systems. We have

THEOREM. *If $D_0, D_1, \ldots$ is a determinant system for $I+T$ of order r, then*

$$D_n\begin{pmatrix} \xi_1 T & \ldots & \xi_n T \\ x_1 & \ldots & x_n \end{pmatrix} = D_n\begin{pmatrix} \xi_1 & \ldots & \xi_n \\ Tx_1 & \ldots & Tx_n \end{pmatrix} \quad \textit{for} \quad n = 0, 1, \ldots \tag{10}$$

Moreover,

$$D_r\begin{pmatrix} \xi_1 T & \ldots & \xi_r T \\ x_1 & \ldots & x_r \end{pmatrix} = (-1)^r D_r\begin{pmatrix} \xi_1 & \ldots & \xi_r \\ x_1 & \ldots & x_r \end{pmatrix}. \tag{11}$$

Since $\zeta_i = -\zeta_i T$ and $z_i = -Tz_i$ $(i = 1, \ldots, r)$, the formulae (11) and (10) for $n = r$ follow from (7). The proof of (10) for $n > r$ is based on the well-known formula

$$\begin{vmatrix} a_{1,1} & \ldots & a_{1,k+r} \\ \multicolumn{3}{c}{\dotfill} \\ a_{k+r,1} & \ldots & a_{k+r,k+r} \end{vmatrix} \tag{12}$$

$$= \sum_{\mathfrak{p}} \operatorname{sgn}\mathfrak{p} \begin{vmatrix} a_{p_1,1} & \ldots & a_{p_1,k} \\ \ldots & \ldots & \ldots \\ a_{p_k,1} & \ldots & a_{p_k,k} \end{vmatrix} \cdot \begin{vmatrix} a_{p_{k+1},k+1} & \ldots & a_{p_{k+1},k+r} \\ \ldots & \ldots & \ldots \\ a_{p_{k+r},k+1} & \ldots & a_{p_{k+r},k+r} \end{vmatrix}$$

$$= \sum_{\mathfrak{q}} \operatorname{sgn}\mathfrak{q} \begin{vmatrix} a_{1,q_1} & \ldots & a_{1,q_k} \\ \ldots & \ldots & \ldots \\ a_{k,q_1} & \ldots & a_{k,q_k} \end{vmatrix} \cdot \begin{vmatrix} a_{k+1,q_{k+1}} & \ldots & a_{k+1,q_{k+r}} \\ \ldots & \ldots & \ldots \\ a_{k+r,q_{k+1}} & \ldots & a_{k+r,q_{k+r}} \end{vmatrix},$$

where the permutations $\mathfrak{p}, \mathfrak{q}$ are the same as in (9).

Therefore, by (7), (12), (11), (5) and well-known properties of classical determinants, we obtain

$$D_{r+k}\begin{pmatrix} \xi_1 T & \ldots & \xi_{r+k} T \\ x_1 & \ldots & x_{r+k} \end{pmatrix}$$

$$= (-1)^r \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p} \operatorname{sgn}\mathfrak{q} \begin{vmatrix} \xi_{p_1} TBx_{q_1} & \ldots & \xi_{p_1} TBx_{q_k} \\ \ldots & \ldots & \ldots \\ \xi_{p_k} TBx_{q_1} & \ldots & \xi_{p_k} TBx_{q_k} \end{vmatrix} \cdot D_r\begin{pmatrix} \xi_{p_{k+1}} & \ldots & \xi_{p_{k+r}} \\ x_{q_{k+1}} & \ldots & x_{q_{k+r}} \end{pmatrix}$$

$$= (-1)^r \times$$

$$\times \sum_{\mathfrak{q}} \operatorname{sgn}\mathfrak{q} \begin{vmatrix} \xi_1 (TB - \sum\limits_{i=1}^{r} z_i \cdot \eta_i) x_{q_1} & \ldots & \xi_1 (TB - \sum\limits_{i=1}^{r} z_i \cdot \eta_i) x_{q_k} & \xi_1 z_1 & \ldots & \xi_1 z_r \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_{k+r} (TB - \sum\limits_{i=1}^{r} z_i \cdot \eta_i) x_{q_1} & \ldots & \xi_{k+r} (TB - \sum\limits_{i=1}^{r} z_i \cdot \eta_i) x_{q_k} & \xi_{k+r} z_1 & \ldots & \xi_{k+r} z_r \end{vmatrix} \times$$

$$\times \begin{vmatrix} \zeta_1 x_{q_{k+1}} & \ldots & \zeta_1 x_{q_{k+r}} \\ \ldots & \ldots & \ldots \\ \zeta_r x_{q_{k+1}} & \ldots & \zeta_r x_{q_{k+r}} \end{vmatrix}$$

$$= (-1)^r \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p} \operatorname{sgn}\mathfrak{q} \begin{vmatrix} \xi_{p_1} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_{q_1} & \ldots & \xi_{p_1} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_{q_k} \\ \ldots & \ldots & \ldots \\ \xi_{p_k} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_{q_1} & \ldots & \xi_{p_1} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_{q_k} \end{vmatrix} \times$$

$$\times D_r\begin{pmatrix} \xi_{p_{k+1}} & \ldots & \xi_{p_{k+r}} \\ x_{q_{k+1}} & \ldots & x_{q_{k+r}} \end{pmatrix}$$

$$= (-1)^r \sum_{\mathfrak{p}} \operatorname{sgn}\mathfrak{p} \begin{vmatrix} \xi_{p_1} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_1 & \ldots & \xi_{p_k} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_{k+r} \\ \ldots & \ldots & \ldots \\ \xi_{p_k} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_1 & \ldots & \xi_{p_k} (BT - \sum\limits_{i=1}^{r} y_i \cdot \zeta_i) x_{k+r} \\ \zeta_1 x_1 & \ldots & \zeta_1 x_{k+r} \\ \ldots & \ldots & \ldots \\ \zeta_r x_1 & \ldots & \zeta_r x_{k+r} \end{vmatrix} \times$$

$$\times \begin{vmatrix} \xi_{p_{k+1}} z_1 & \ldots & \xi_{p_{k+1}} z_r \\ \ldots & \ldots & \ldots \\ \xi_{p_{k+r}} z_1 & \ldots & \xi_{p_{k+r}} z_r \end{vmatrix}$$

$$= \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn} \mathfrak{p} \operatorname{sgn} \mathfrak{q} \begin{vmatrix} \xi_{p_1} BTx_{q_1} & \dots & \xi_{p_1} BTx_{q_k} \\ \dots & \dots & \dots \\ \xi_{p_k} BTx_{q_1} & \dots & \xi_{p_k} BTx_{q_k} \end{vmatrix} \cdot D_r \begin{pmatrix} \xi_{p_{k+1}} & \dots & \xi_{p_{k+r}} \\ Tx_{q_{k+1}} & \dots & Tx_{q_{k+r}} \end{pmatrix}$$

$$= D_{r+k} \begin{pmatrix} \xi_1 & \dots & \xi_{r+k} \\ Tx_1 & \dots & Tx_{r+k} \end{pmatrix}.$$

This completes the proof.

Having proved the above theorem we can easily obtain the formulae for solutions of the Fredholm type (see R. Sikorski [3]).

REFERENCES

[1] A. Grothendieck, *La théorie de Fredholm*, Bulletin de la Société Mathématique de France (1956), p. 319-384.

[2] R. Sikorski, *Determinant systems*, Studia Mathematica 18 (1959), p. 161-186.

[3] — *Remarks on Leżański's determinants*, ibidem 20 (1961), p. 145-161

Reçu par la Rédaction le 19. 9. 1962

COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 2

SUR LES ÉQUATIONS HYPERBOLIQUES AVEC PETIT PARAMÈTRE

PAR

J. KISYŃSKI (VARSOVIE)

Considérons le problème de Cauchy abstrait

$$(0.1) \qquad \varepsilon x''(t) + x'(t) + Ax(t) = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

avec petit $\varepsilon > 0$ et opération A autoadjointe non négative dans un espace de Hilbert H_0. Dans le cas où $H_0 = (-\infty, +\infty)$, A est l'opération de multiplication par un nombre (non négatif) et, pour $0 \leqslant t < +\infty$, la solution du problème (0.1) prend alors la forme

$$(0.2) \qquad x(t) = x_0(t) + \varepsilon e^{-t/\varepsilon}(Ax_0 + x_1) + y_\varepsilon(t),$$

où $\lim_{\varepsilon \to +0} y_\varepsilon(t) = \lim_{\varepsilon \to +0} y'_\varepsilon(t) = 0$ uniformément sur l'intervalle $0 \leqslant t < +\infty$ et $x_0(t) = e^{-tA}x_0$ est la solution du problème

$$(0.3) \qquad x'(t) + Ax(t) = 0, \quad x(0) = x_0.$$

Zlámal [8]-[10] a montré par la méthode de Fourier que le comportement asymptotique du type (0.2) a aussi lieu pour les équations hyperboliques du second ordre du type (0.1) dans lesquelles $-A$ est une opération symétrique fortement elliptique (1). Ici, je me propose de montrer que le comportement asymptotique du type (0.2) a lieu pour tout problème du type (0.1) avec une opération A autoadjointe non négative dans un espace de Hilbert quelconque.

1. Semi-groupes à un paramètre d'opérations. Rappelons d'abord quelques notions de la théorie des semi-groupes.

Soit X un espace de Banach. On appelle *semi-groupe* (ou *groupe* respectivement) à un paramètre d'opérations dans l'espace X toute famille $S(t)$ où $0 \leqslant t < +\infty$ (ou $-\infty < t < +\infty$ respectivement) d'opé-

(1) Les équations hyperboliques dans lesquelles la seconde dérivée au temps contient un petit paramètre ont été étudiées dans [3] et dans la thèse de Nikolski citée dans [2]. Dans [8] et [10], il s'agit des équations aux coefficients dépéndant de t.