COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 1

ON BASES WITH RESPECT TO A CLOSURE OPERATOR WITH THE EXCHANGE PROPERTY

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It is known that the classical theorem on the existence of bases in vector spaces is a particular case of a more general proposition: if C is a closure operator in X, having a finite character and the so-called exchange property, then there exists a C-basis of X (see below 1 (iii) and 3 (i)).

This paper contains some remarks on closure operators with the exchange property and, in particular, a theorem on the existence of a basis, in which the hypothesis of a finite character is replaced by a weaker one (Theorem 1). I prove also a converse of this theorem for the case of a topological space (Theorem 2).

Some modifications of my primary proofs are due to Professor E. Marczewski.

1. Closure operators. Let us recall basic properties of closure operators and some related notions.

By *closure operator* in a fixed set X we mean every extensive, monotone and idempotent function C which associates a subset $C(E)$ of X with each subset E of X (cf. e. g. Birkhoff [1], p. 49, Schmidt [4] and [5]). In the sequel the letter C will always denote a closure operator.

If $G\subset E\subset X$ and $C(G)\supset E$, we say that G *C-generates* E, or that G is *a set of C-generators of* E, or else, that G is *C-dense in* E. A set is called *finitely C-generated* if it is C-generated by a finite subset.

We say that I is *C-independent*, or *C-isolated* (Schmidt [4], p. 38) if I is a minimal set of generators of $C(I)$, or, in other words, if $a\notin C(I\setminus\{a\})$ for each $a\in I$. Each C-independent set of C-generators of E is called a *C-basis* of E. It is easy to see that

(i) *Each C-basis of E is a maximal C-independent subset of E.*

(ii) *Each finite set of C-generators of E contains a finite C-basis.*

The converse of (i) is not generally true. Moreover bases do not always exist.

By definition, a closure operator C in X has *a finite character*, whenever for each $E \subset X$

$$C(E) = \bigcup C(F),$$

where F runs over all finite subsets of E.

Kuratowski-Zorn lemma implies the following proposition (see e. g. Schmidt [4], p. 39).

(iii) *If C has a finite character, then every C-independent subset of $E \subset X$ is contained in a maximal C-independent subset of E.*

The hypothesis of a finite character is essential.

2. Relativization and localization. If C is a closure operator in X and if $Y \subset X$, then we put $C_Y(E) = C(E) \cap Y$ for any $E \subset Y$ (cf. Kuratowski [3], p. 22). It is easy to verify that

(i) *C_Y is a closure operator in Y.*

(ii) *If C has a finite character, so has C_Y.*

(iii) *For subsets of Y the notions of independence, generators, and bases, are equivalent for C and C_Y, respectively.*

We say that a is a *C-independent* or else *C-isolated point of E*, in symbols: $a \in \mathrm{Ind}(E, C)$, whenever $a \notin C(E - \{a\})$. Obviously, a set I is C-independent if and only if $\mathrm{Ind}(I, C) = I$. In view of (iii) we have $\mathrm{Ind}(E, C) = \mathrm{Ind}(E, C_E)$.

We say that the operator C in X has *a finite character in $a \in X$*, in symbols: $a \in \mathrm{Fin}(C)$, whenever for every set $E \subset X$ such that $a \in C(E)$ there exists a finite set $F \subset E$ such that $a \in C(F)$. Obviously, C has a finite character if and only if $\mathrm{Fin}(C) = X$. It is easy to prove that

(iv) $\mathrm{Ind}(X, C) \subset \mathrm{Fin}(C)$.

The following proposition is essential for the sequel:

(v) *If $K \subset \mathrm{Fin}(C)$, then C_K has a finite character.*

It is sufficient to prove that

$$C_K(E) \subset \bigcup C_K(F),$$

where E is any subset of K and F runs over all finite subsets of E. Let us suppose $a \in C_K(E)$. Hence

$$a \in K \cap C(E) \subset \mathrm{Fin}(C) \cap C(E).$$

Consequently, there exists, by definition of the set $\mathrm{Fin}(C)$, a finite set $F \subset E \subset K$ such that $a \in C(F)$. Since $a \in C_K(E) \subset K$, we have $a \in C(F) \cap K = C_K(F)$, q. e. d.

3. Exchange property and bases. We suppose in this section that C is a closure operator in X, with *the exchange property* (cf. e. g. Schmidt [5]), i. e. that for any $E \subset X$, and $a, b \in X$, the relations $a \notin C(E)$ and $a \in C(E \cup \{b\})$ imply $b \in C(E \cup \{a\})$. It is known that, under the above hypothesis (see e. g. Schmidt [5], p. 236):

(i) *If the set I is C-independent and $I \cup \{a\}$ is non C-independent, then $a \in C(I)$,*

whence

(ii) *Each maximal C-independent subset of $E \subset X$ is a basis of E.*

Now let us remark that obviously

(iii) *If $Y \subset X$, and C is a closure operator in X having the exchange property, then C_Y has also this property.*

Next we shall prove that

(iv) *Each C-independent subset of $\Phi = \mathrm{Fin}(C)$ is a subset of a C-basis of Φ.*

In view of 2 (iii), it is sufficient to prove that each C_Φ-independent subset of Φ is a subset of a C_Φ-basis of Φ.

In view of 2 (v), the closure operator C_Φ has a finite character, whence, for each C_Φ-independent subset I of Φ, there exists by 1 (iii) a maximal C_Φ-independent subset B of Φ, containing I. By (ii) and (iii) the set B is a C_Φ-basis of Φ, q. e. d.

The following theorem is an easy consequence of (iv):

THEOREM 1. *If C is a closure operator in X having the exchange property, and the set $\mathrm{Fin}(C)$ generates X, then each C-independent subset I of $\mathrm{Fin}(C)$ is contained in a C-basis of X.*

Obviously it is sufficient to put $I = 0$ in order to obtain the existence of a basis of X under the hypothesis of theorem 1.

Let us remark incidentally that, for closure operators in X with the exchange property,

(v) *If $Y \subset X$ is finitely C-generated, then every subset $Z \subset Y$ is also finitely C-generated,*

whence

(vi) *If X is finitely C-generated, then C has a finite character.*

In order to prove (v) let us recall the following proposition proved under the hypothesis of the exchange property (see e. g. Bleicher-Marczewski [2], p. 210, proposition (i′)):

(vii) *If I is a finite C-independent set and G is a set of C-generators of I, then $|G| \geqslant |I|$, where $|\ |$ denotes the number of elements.*

Let us denote by G a finite set of generators of Y. If I is a C-independent subset of Y, then, by (vii), we have $|I| \leqslant |G|$. Consequently, there exists a C-independent subset J of Z having the greatest number of elements. J is hence a maximal C-independent subset of Z and, by (ii), a C-basis of Z. Since $|J| \leqslant |G|$, the set Z is finitely C-generated. The proposition (v) is thus proved.

4. Topological closure. A closure operator C in X is called *topological* whenever 1) it is additive, i. e. $C(A \cup B) = C(A) \cup C(B)$ for any subsets A and B of X, 2) $C(\{a\}) = \{a\}$ for each $a \in X$. In other words, C is a topological closure operator if and only if the set X with the closure operator satisfies the well-known axioms of Kuratowski ([3], p. 20).

It is easy to see that

(i) *Each topological closure operator has the exchange property,*

and

(ii) *For any topological closure operator we have*

$$\mathrm{Ind}(X, C) = \mathrm{Fin}(C),$$

which follows from 2(iv) and from the equality $C(F) = F$, valid for every finite set F, whenever C is topological.

Let us prove the following lemma:

(iii) *If C is a topological closure operator in X, the set G is C-dense in X, and $G \setminus \{p\}$ is not, then p is C-isolated in X.*

Since

$$C(G-\{p\}) \cup \{p\} = C(G-\{p\}) \cup C(\{p\}) = C(G) = X,$$

and

$$C(G-\{p\}) \neq X,$$

we have $p \notin C(G \setminus \{p\})$. Consequently $\{p\}$ is an open set, q. e. d.

Proposition (iii) implies the following theorem, containing, in view of (ii), the converse of Theorem 1 in the case of topological space.

THEOREM 2. *If C is a topological closure operator in X and B a C-basis* (1) *of X, then B is the set of all C-isolated points of X (in symbols: $B = \mathrm{Ind}(X, C)$) and B is C-dense in X.*

In view of (i), Theorems 1 and 2 give the following equivalence:

If C is a topological closure operator in X, then there exists a C-basis B of X if and only if the set $\mathrm{Ind}(X, C)$ of all C-isolated points of X is C-dense in X. We have then $B = \mathrm{Ind}(X, C)$.

Let us remark that Theorem 2 is not generally true without the hypothesis that the operator C is topological. In fact, in the first example from the quoted paper by Bleicher and Marczewski [2] (p. 210), two disjoint sets I and J are C-bases of X, while $\mathrm{Fin}(C) = 0 = C(0)$. Consequently 1° $\mathrm{Fin}(C)$ does not C-generate X and 2° there exist different C-bases of X.

(1) Not to be confused with a basis in the topological sense.

REFERENCES

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Reçu par la Rédaction le 1. 5. 1962