cette équation. Reste à étudier le cas où il existe deux nombres m_0 et n_0 (où $m_0 < n_0$) tels que $g_{m_0}(x) = g_{n_0}(x)$. Alors $g_{n_0}(x) = g_{n_0-m_0}[g_{m_0}(x)]$, d'où $g_{n_0-m_0}(x) = x$. On a ensuite $f[g_{n_0-m_0-1}(x)] = g_{n_0-m_0}(x) = x$. La fonction $f(x)$ est donc inverse de $g_{n_0-m_0-1}(x)$ et par suite elle est univalente. Comme continue, elle est strictement monotone. Il existe donc en vertu du corollaire qui précède une infinité de fonctions $g \in C_{\langle 0,1 \rangle}$ satisfaisant à (1).

Exemple. Soit $\eta \in \langle 0,1 \rangle$ un nombre irrationnel. Posons

$$f(x) = \begin{cases} x+\eta & \text{pour} \quad 0 \leqslant x \leqslant 1-\eta, \\ x+\eta-1 & \text{pour} \quad 1-\eta < x < 1, \\ 0 & \text{pour} \quad x = 1. \end{cases}$$

La fonction f transforme donc l'intervalle $\langle 0,1 \rangle$ en lui-même, mais elle est discontinue en deux points. Nous allons montrer que la seule fonction $g \in C_{\langle 0,1 \rangle}$ qui satisfait à l'équation (1) est la fonction $g = e$. En posant $\xi_0 = g(\xi_0)$, (1) entraîne en effet $f(\xi_0) = g[f(\xi_0)]$. Le point $\xi_1 = f(\xi_0)$ est donc un point fixe de la transformation g. En posant $\xi_1 = f(\xi_{n-1})$, on constate par récurrence que tous les nombres ξ_n sont des points fixes de la transformation g. Or l'ensemble des nombres ξ_n étant dense dans $\langle 0,1 \rangle$, la fonction continue g, dont l'ensemble des points fixes est dense, est nécessairement la fonction-identité.

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DIFFERENTIABILITY OF MONOTONIC FUNCTIONS

BY

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This paper gives still another proof of the theorem of Lebesgue that a non-decreasing function $f(x)$ on a closed interval $[a, b]$ has a finite derivative $f'(x)$ almost everywhere. Riesz [1], p. 5-9, has proposed an elegant elementary proof that uses no measure theory beyond that of sets of measure zero. The proof is elegant and simple in the case where $f(x)$ is continuous, but the details ([2], p. 69-75) of the straightforward extension of Riesz's proof to the discontinuous case are troublesome and tedious. We use only elementary methods, borrowing half of Riesz's proof. Our proof for the general case is then no longer than Riesz's proof for the continuous case.

Recently in this journal, Boas [3] gave a simple proof, but one that required some measure-theoretic preliminaries, that if f is a jump function, then f' exists and is zero almost everywhere. Since each monotonic function is the sum of a monotonic continuous function and a jump function, the result for arbitrary monotonic function follows from the result for continuous functions and for jump functions. There shortly followed a completely elementary proof for jump functions by Lipiński [4]. We end this paper with a short proof that f' is zero almost everywhere if f is a jump function.

The principal innovation in our proof of Lebesgue's theorem is the idea of studying a continuous inverse of $f(x)$ to make the extension to the discontinuous case in a painless way. We take as our starting point Lemma 1, which is the second part of what Riesz proved in detail.

Lemma 1. *If $F(y)$ is continuous and non-decreasing on $[A, B]$, then $F'(y) \leqslant +\infty$ exists almost everywhere.*

The first part of Riesz's proof shows that $F'(y) < +\infty$ almost everywhere. We do not need this fact in Lemma 1, and will give a separate proof later that the derivative is finite almost everywhere even in the discontinuous case.

Lemma 2. *Let $f(x)$ be a strictly increasing function on $[a, b]$. Then $f(x)$ has a continuous inverse; that is, there exists a continuous, non-de-*

creasing function $F(y)$ defined on $[f(a), f(b)]$ such that $F(f(x)) = x$ for each $x \in [a, b]$.

Geometrically, the construction of F is evident. An analytical expression for F is

$$F(y) = \sup\{t: f(t) \leqslant y\}. \tag{1}$$

It is obvious that F is non-decreasing. Hence, to show that F is continuous, it is enough to show that the range of F is all of $[a, b]$. This follows from the trivially verified fact that $F(f(x)) = x$.

Now let $f(x)$ be a non-decreasing, but possibly discontinuous function on $[a, b]$. We must show that $f'(x) < +\infty$ exists almost everywhere. Without loss of generality, we may suppose that f is strictly increasing, and satisfies, moreover, the condition

$$f(y) - f(x) \geqslant y - x \quad \text{whenever} \quad y \geqslant x, \tag{2}$$

since we could otherwise consider $f(x) + x$. If $F(y)$ is the function of Lemma 2, then by Lemma 1, $F'(y) \leqslant +\infty$ exists almost everywhere. We write

$$\frac{f(y)-f(x)}{y-x} = \frac{f(y)-f(x)}{F(f(y))-F(f(x))} = \left(\frac{F(f(y))-F(f(x))}{f(y)-f(x)}\right)^{-1}. \tag{3}$$

Thus, for every point of continuity of $f(x)$ such that $f(x)$ does not lie in the exceptional set E_f where F' fails to exist, we see that $f'(x) \leqslant +\infty$ exists. But the set of points of discontinuity of f is countable at most. Furthermore, $f^{-1}(E_f)$ has measure zero, since if I is any interval, then $f^{-1}(I)$ is a union of intervals of total length not exceeding the length of I, by (2). Hence $f'(x) \leqslant +\infty$ exists almost everywhere.

To complete the proof, we need only show that the set E_∞, of those x for which $f'(x) = +\infty$, has measure zero. Our method is close to one used by Lipiński [4]. For $C > 0$, let E_C be the set of those $x \in (a, b)$ for which there exist $s = s_x$ and $t = t_x$ with $s < x < t$ such that

$$f(t) - f(s) > C(t-s). \tag{4}$$

It is clear that E_C is open, and hence the disjoint union of open intervals:

$$E_C = \bigcup (a_n, b_n). \tag{5}$$

Let us denote by $[a_n', b_n']$ any closed sub-interval of (a_n, b_n) such that

$$b_n' - a_n' = \tfrac{1}{2}(b_n - a_n). \tag{6}$$

Now it is clear that $[a_n', b_n']$ is covered by the open intervals (s_x, t_x) for $x \in [a_n', b_n']$ and that $(s_x, t_x) \subset (a_n, b_n)$. By the Heine-Borel theorem, there is a finite subcovering, say (s_k, t_k) where $k = 1, 2, \ldots, N$; and if we choose a subcovering with N as small as possible, then each point of $\bigcup (s_k, t_k)$ lies in at most two of the intervals (s_k, t_k), because given any three open intervals with a common point, some one is contained in the union of the other two.

Hence, using (4), we have

$$b_n' - a_n' \leqslant \sum (t_k - s_k) \leqslant C^{-1} \sum \{f(t_k) - f(s_k)\} \leqslant 2C^{-1}\{f(b_n) - f(a_n)\}, \tag{7}$$

and then by (6) and (7) we have

$$\sum (b_n - a_n) \leqslant 4C^{-1} \sum \{f(b_n) - f(a_n)\} \leqslant 4C^{-1}\{f(b) - f(a)\}. \tag{8}$$

Since E_∞ is contained in each E_C, it follows, on letting $C \to \infty$, that E_∞ has measure zero, and the proof is complete.

We now show briefly that if f is a jump function, then $f' = 0$ almost everywhere. To say that f is a jump function is to say that $f(x) = \sum f_k(x)$ where $f_k(x) = 0$ for $x < a_k$, $f_k(x) = S_k$ for $x > a_k$, $0 \leqslant f_k(a_k) \leqslant S_k$, and $\sum S_k < \infty$. Choose $C > 0$, then $\varepsilon > 0$, and let $g = \sum' f_k$, where $\sum'$ is a sum over a finite set of indices k such that $\sum S_k - \sum' S_k < \varepsilon C$. Then $h = f - g$ is non-decreasing, $h(b) - h(a) < \varepsilon C$, and $h' = f'$ except on finitely many points. By (8), we see that $\{x: h'(x) \geqslant C\}$ can be covered by open intervals whose total length is less than 4ε. Hence $\{x: f'(x) \geqslant C\}$ has measure zero for each positive C. Since $\{x: f'(x) > 0\} = \bigcup_n \{x: f'(x) \geqslant 2^{-n}\}$, the result is proved.

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