rems are modified to allow $\mathcal{I}$ to be a σ-ideal. The following example provides the negative answer.

Let X consist of points (x, y) of the closed unit square in the Euclidean plane with $(0, 0)$ as the lower left-hand vertex. Define the metric ϱ on X by

$$\varrho\{(x_1, y_1), (x_2, y_2)\} = \begin{cases} |x_2 - x_1| & \text{if} \quad y_1 = y_2, \\ 2 & \text{if} \quad y_1 \neq y_2. \end{cases}$$

The reader can verify that X is a non-separable metric space. Let R be the set of all points whose abscissae are rational and let f be the characteristic function of R. Further, let $\mathcal{I}$ be the class of all countable sets and put $a = 1$. The reader can verify that the modified conditions of theorems I and II are satisfied. Yet the conclusions of these theorems do not hold.

2. The question arises whether the conclusion of theorem III holds if the hypotheses of that theorem are modified to make X 1-separable and $\mathcal{I}$ a σ_1-ideal. The following example shows that it does not.

Let X be the Cartesian product of the closed unit interval and the set of all countable ordinals. Thus, the points of X are of the form (x, β) where $0 \leqslant x \leqslant 1$ and $0 \leqslant \beta < \omega_1$. Define the following metric ϱ on X:

$$\varrho\{(x_1, \beta_1), (x_2, \beta_2)\} = \begin{cases} |x_2 - x_1| & \text{if} \quad \beta_1 = \beta_2, \\ 2 & \text{if} \quad \beta_1 \neq \beta_2. \end{cases}$$

Let Y be the real line and let $\mathcal{I}$ consist of $\emptyset$ alone.

Now for each countable ordinal number η choose a real function f_η of Baire class η defined on $[0,1]$. Then, define f on X by

$$f(x, \beta) = f_\beta(x).$$

Since $\mathcal{I}$ consists of $\emptyset$ alone, (a. e.) means everywhere, and $\mathcal{I}$ is a σ_1-ideal. The reader will verify further that X is 1-separable, that f has the property Δ and that f is not a Borel-measurable function on X.

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*TO WACŁAW SIERPIŃSKI
ON HIS 80-TH BIRTHDAY*

ON SOME PROPERTIES OF HAMEL BASES

BY

P. ERDÖS (BUDAPEST)

I dedicate this little note to Professor Wacław Sierpiński since I use in it methods which he used very successfully on so many occasions.

Throughout this paper $\alpha, \beta, \gamma, \ldots$ will denote ordinal numbers, $n_i, n_\alpha, \ldots$ integers, $r_\alpha, \ldots$ rational numbers, $r_\alpha^+, \ldots$ non-negative rationals and $a, a_\alpha, b, \ldots$ real numbers. H will denote a Hamel basis of the real numbers, H^* the set of all numbers of the form $\sum_\alpha n_\alpha a_\alpha$ $(a_\alpha \in H)$ (the sum is finite) and H^+ the set of all numbers of the form $\sum_\alpha r_\alpha^+ a_\alpha$ $(a_\alpha \in H)$. Measure will always be the Lebesgue measure, and (a, b) will denote the set of numbers $a < x < b$.

Sierpiński showed [1] that there are Hamel bases of measure 0 and also Hamel bases which are not measurable.

We are going to prove the following theorems:

THEOREM 1. *H^* is always non-measurable. In fact H^* has inner measure 0 and for every (a, b) the outer measure of $H^* \frown (a, b)$ is $b - a$.*

THEOREM 2. *Assume $c = \aleph_1$. Then there is an H for which H^+ has measure 0.*

Proof of Theorem 1. The sets $H^* + 1/n$, $2 \leqslant n < \infty$, are pairwise disjoint. Thus a simple argument shows that H^* has inner measure 0.

For every x there exists an n_x so that $n_x \cdot x$ is in H^*, or the sets $1/nH^*$, $2 \leqslant n < \infty$, cover the whole interval $(-\infty, +\infty)$. Hence H^* cannot have outer measure 0, and thus by the Lebesgue density theorem it has a point, say x_0, of outer density 1. But then (since H^* is an additive group) every point of $x_0 + H^*$ is a point of outer density 1 of H^*. Finally, it is easy to see that H^* is everywhere dense (since, if a and b are rationally independent, the numbers $n_1 a + n_2 b$ are everywhere dense).

Now it is easy to deduce that the outer measure of $H^* \frown (a, b)$ is $b - a$. To see this observe that since H^* has outer density 1 at x_0, for every $\varepsilon > 0$ there exist arbitrarily small values of η, such that the outer measure of $H^* \frown (x_0 - \eta, x_0 + \eta)$ is greater than $2(1-\varepsilon)\eta$; but consequently the same holds for $H^* \frown (x_0 + t - \eta, x_0 + t + \eta)$, where t is an arbitrary element of H^*. Since H^* is everywhere dense, a simple argument shows that the outer measure of $H^* \cap (a, b)$ is greater than $(1-\varepsilon)(b-a)-3\eta$. Since this holds for every ε and η, the outer measure is $b-a$, which completes the proof of Theorem 1.

Now we prove Theorem 2. In fact we shall prove a somewhat stronger theorem:

THEOREM 2'. *Assume* $c = \aleph_1$. *Then there is an* H *such that* H^+ *is a Lusin set (see* [2], *p.* 36-37), *i. e. it intersects every nowhere dense perfectset in a set of power* $\leqslant \aleph_0$.

It is well known (and easy to see) that such a set has the property that if ε_k, $1 \leqslant k < \infty$, is any sequence of numbers, it can be covered by intervals I_k of length ε_k $(1 \leqslant k < \infty)$ (see [3] and also [2], p. 37-39).

We shall construct our H by transfinite induction. Let $\{F_a\}$, $1 \leqslant a < \Omega_1$, be the set of all nowhere dense perfect sets (as is well known, there are $c = \aleph_1$ perfect sets) and let x_a, $1 \leqslant a < \Omega_1$, be a well-ordering of the set of all real numbers. Put

$$F^{(a)} = \bigcup_{1 \leqslant \gamma < a} F_\gamma .$$

$F^{(a)}$ is a set of the first category and for $a > \gamma$ we have $F^{(a)} \supset F^{(\gamma)}$.

We shall denote by $\{a_a\}$, $1 \leqslant a < \Omega_1$, the elements of H. Assume that for $a < \beta$ the a_β have already been constructed. We choose a_β and $a_{\beta+1}$ as follows: Let x_δ be the x_a of smallest index which is not of the form $\sum_i r_{a_i} a_{a_i}$, $a_i < \beta$. Put

$$x_\delta = u - v, \tag{1}$$

where u and v have the following properties:

I. $\{u, v, a_a\}$, $1 \leqslant a < \beta$, are rationally independent.

II. The numbers

$$r_1 u + r_2 v + \sum_i r_{a_i} a_{a_i}, \qquad a_i < \beta, \tag{2}$$

are never in $F^{(\beta)}$, unless $r_1 = -r_2 \neq 0$.

Then put $a_\beta = v$ and $a_{\beta+1} = u$. First we show that such values u and v exist.

Put $u = v + x_\delta$. Then II is equivalent to the relation

$$\Big((r_1+r_2)v + r_1 x_\delta + \sum_i r_{a_i} a_{a_i}\Big) \notin F^{(\beta)}$$

for every choice of $r_1 + r_2 \neq 0$ and arbitrary r_{a_i}, a_{a_i}, $a_i < \beta$. Thus v is in none of the sets

$$\Big(F^{(\beta)} - \sum_i r_{a_i} a_{a_i} - r_1 x_\delta\Big) / (r_1 + r_2). \tag{3}$$

Clearly all sets (3) are sets of the first category and there are only $\aleph_0$ of them. Thus their union is also of the first category and hence there

exists a set of v's of second category which is not contained in their union and which thus satisfies II. It is easy to see that there exists at most a countable number of choices of v and $u = v + x_\delta$ which do not satisfy I; hence there exist u and v satisfying both I and II.

This construction can clearly be carried out for all ordinal numbers $\beta < \Omega_1$, and, since $c = \aleph_1$, it gives a Hamel-base H. Clearly H^+ is a Lusin-set. To see this it is sufficient to show that $H^+ \cap F^{(a)}$ has for every $a < \Omega_1$ a power not exceeding $\aleph_0$. Let $\sum_{i=1}^{i} r^+_{\xi_i} a_{\xi_i}$ $(\xi_1 < \ldots < \xi_t)$ be an element of H^+. Since $c = \aleph_1$, there are only denumerably many elements of H^+ with $\xi_t \leqslant a$. If $\xi_t > a$, then by our construction $\sum_{i=1}^{t} r^+_{\xi_i} a_{\xi i}$ is not in $F^{(a)}$ since, by II, if $\xi_t > a$, then $\sum_{i=1}^{t} r_{\xi_i} a_{\xi_i}$ can be in $F^{(a)}$ only if $\xi \quad +1 = \xi_t$ and $r_{\xi_{t-1}} = -r_{\xi_t}$, but it is then not in H^+. This completes the proof of Theorem II.

We have really proved the following stronger statement:

There exists a Hamel-base H *with a well-ordering* $\{a_a\}$ *such that the set of real numbers* $\sum_{i=1}^{t} r_{a_i} a_{a_i}$ *for which*

$$a_{t-1} \neq a_t - 1 \quad \text{or} \quad r_{a_t} + r_{a_{t-1}} = r_{a_t} + r_{a_{t-1}} \neq 0$$

is a Lusin set.

Kuczma asked in [4] the following question: Let $f(X+Y) = f(X) + f(Y)$ and assume that $f(Z) < c$ for every $Z \in P$, where P is such a set that every real number can be written in the form $Z_1 - Z_2$, $Z_1, Z_2 \in P$. Does it follow then that $f(X) = cX$? The answer is negative. To see this let $f(a_a) \leqslant 0$ for every $a_a \in H$, let $f(a_a)$ be non-linear and let us extend $f(X)$ for every real X by $f(u+v) = f(u) + f(v)$. Clearly $f(Z) \leqslant 0$ for every $Z \in H^+$, every real number is of the form $Z_1 - Z_2$, $Z_1, Z_2 \in H^+$, and $f(X) \neq cX$.

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