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SOME THEOREMS ON BOREL-MEASURABLE FUNCTIONS

BY

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INTRODUCTION

The subject of this paper arose from a problem due to Professor E. Marczewski (New Scottish Book, Problem 277). In solving it I noticed that more general results are obtainable. These are contained in [2]. Further generalisations were obtained by Traczyk [3]. He generalised my results on real functions of a real variable to the case of mappings of separable metric spaces into separable and complete metric spaces. Independently, I generalised my earlier results to the case of real functions on a general metric space. The final generalisations (combining Dr. Traczyk's results and mine) are contained in theorems I and II of this paper. From these a new result contained in theorem III is obtained.

The axiom of choice is assumed throughout this work.

I wish to emphasize that it was Professor Marczewski's problem that gave me the original incentive. I also wish to thank him and Dr. Traczyk for a valuable correspondence drawing my attention to Dr. Traczyk's work.

NOTATION AND DEFINITIONS

In what follows all functions map a general metric space X into a separable and complete metric space Y.

"Iff" will stand for "if and only if". The empty set will be denoted by $\emptyset$. The cardinal number of any set S will be denoted by $|S|$.

Given any ordinal number γ, a class of sets $\mathscr{E}$ is called a σ_γ-*ideal* iff it satisfies the following conditions:

(i) If $E_1 \in \mathscr{E}$ and $E_2 \subset E_1$, then $E_2 \in \mathscr{E}$.

(ii) Let T be a set of indices such that $|T| \leqslant \aleph_\gamma$ and $E_t \in \mathscr{E}$ whenever $t \in T$. Then

$$\bigcup_{t \in T} E_t \in \mathscr{E}.$$

Clearly, a σ_0-ideal means a σ-ideal.

X is said to be *γ-separable* iff γ is an ordinal number for which an everywhere dense subset S of X exists such that $|S| \leqslant \aleph_\gamma$. Clearly, 0-separability means separability.

In what follows subsets of X will be referred to as sets.

$\mathcal{I}$ will denote a σ_γ-ideal of sets. A proposition P will be said to "hold (a. e.)" in a set E iff those points of E at which P does not hold form a set belonging to $\mathcal{I}$. "P holds (a. e.)" will mean that P holds (a. e.) in X.

Given any countable ordinal number α, $\mathcal{G}(\alpha)$ will denote the α-additive class of Borel sets and $\mathcal{B}(\alpha)$ the α-class of Borel-measurable functions (see [1], p. 250-305).

The distance between two points y_1 and y_2 in Y will be denoted by $|y_1 - y_2|$.

In the following two definitions X, T and the ordinal number α ($0 \leqslant \alpha < \omega_1$) are fixed. A function f is said *to have the property $D(\alpha)$* with respect to a closed set F at a point x in F, iff for any positive ε there is a neighbourhood G of x and a function g in $\mathcal{B}(\alpha)$ such that $|f(x)-g(x)| < \varepsilon$ (a. e.) in $G \cap F$. The function f is said *to have the property $\Delta(\alpha)$*, iff for any non-empty, closed set F there is some x in F such that f has the property $D(\alpha)$ w. r. t. F at x.

In the following definition X and T are fixed, but α is not. The function f is said *to have the property Δ*, iff for any non-empty, closed set F there is some x in F and some α ($0 < \alpha < \omega_1$) such that f has the property $D(\alpha)$ w. r. t. F at x.

Let $(\alpha_0, \alpha_1, \ldots, \alpha_\eta, \ldots)$ be a transfinite sequence where α_η is defined iff $\eta < \lambda$. Then λ is called the *length* of the transfinite sequence.

STATEMENT OF THE THEOREMS

THEOREM I. *Let X be any γ-separable metric space, $\mathcal{I}$ any σ_γ-ideal of sets and α any fixed ordinal number such that $0 < \alpha < \omega_1$. Then, if the function f has the property $\Delta(\alpha)$, there is for any positive ε a member h of $\mathcal{B}(\alpha)$ such that*

$$|f(x)-h(x)| < \varepsilon \ (a. e.).$$

THEOREM II. *Under the hypotheses of Theorem* I, *there is a member g of $\mathcal{B}(\alpha)$ such that $f(x) = g(x)$ (a. e.).*

THEOREM III. *Let X be any separable metric space and $\mathcal{I}$ any σ-ideal of sets. Then, if the function f has the property Δ, there is a member g of $\mathcal{B}$ such that $f(x) = g(x)$ (a. e.).*

Remarks. For any γ, the class of sets $\mathcal{I}$ consisting of $\emptyset$ alone is a σ_γ-ideal. Thus, if $\mathcal{I}$ consists of $\emptyset$ alone, (a. e.) means everywhere.

It will be shown that in the statement of theorem III the separable metric space X cannot be replaced by a general metric space.

PROOF OF THE THEOREMS

We shall assume that X is infinite (else the theorems hold trivially).

Proof of Theorem I. Assume the hypotheses of theorem I to hold. Let $\mathcal{R}$ be a base of neighbourhoods in X such that $|\mathcal{R}| = \aleph_\gamma$. Well ordering $\mathcal{R}$ we obtain a transfinite sequence $(R_1, R_2, \ldots)$ of length ω_γ. Denote by N the set of all ordinal numbers n for which there is a function h_n in $\mathcal{B}(\alpha)$ such that

$$|f(x)-h_n(x)| < \varepsilon \ \text{(a. e.) in } R_n.$$

The set N is obviously infinite and its elements can be arranged in an increasing transfinite sequence $(n_1, n_2, \ldots)$ of length λ. Clearly $\lambda \leqslant \omega_\gamma$.

We shall prove now that $X = \bigcup_{k<\lambda} R_{n_k}$.

Suppose that it is not true. Put

$$F = X - \bigcup_{k<\lambda} R_{n_k}.$$

Then since F is non-empty and closed, there exists by hypothesis a point x_0 in F, an element R_m of $\mathcal{R}$ containing x_0 and a function h_m in $\mathcal{B}(\alpha)$ such that

$$|f(x)-h_m(x)| < \varepsilon \ \text{(a. e.) in } R_m.$$

Thus

$$R_m \subset \bigcup_{k<\lambda} R_{n_k}.$$

Hence $x_0 \in F \cap \big(\bigcup_{k<\lambda} R_{n_k}\big)$. This is absurd. We conclude that

$$X = \bigcup_{k<\lambda} R_{n_k}.$$

Next put $R_1 = S_1$ and for $k > 1$

$$R_{n_k} - \bigcup_{i<k} R_{n_i} = S_k.$$

Then $X = \bigcup_{k<\lambda} S_k$, where the elements of the union are mutually disjoint and are all differences of open sets. Thus, since $\alpha > 0$, $S_k \in \mathcal{G}(\alpha)$ for each k.

Define now the function h as follows:

$$h(x) = h_{n_k}(x) \quad \text{for} \quad x \in S_k.$$

We observe that, for any k, $h(x) = f(x)$ (a. e.) in S_k. Hence, since $\lambda \leqslant \omega_\gamma$ and $\mathcal{I}$ is a σ_λ-ideal,

$$h(x) = f(x) \ \text{(a. e.)}.$$

It remains to show that $h(x) \in \mathcal{B}(\alpha)$.

Choose any open set G of Y. Then, for any k, since $h_{n_k} \in \mathscr{B}(\alpha)$, $h_{n_k}^{-1}(G) \in \mathscr{B}(\alpha)$. Also, since $S_k \in \mathscr{G}(\alpha)$, we have for each k:

$$S_k \cap h_{n_k}^{-1}(G) \in \mathscr{G}(\alpha).$$

Now $h^{-1}(G) = \bigcup_{k<\lambda} [S_k \cap h_{n_k}^{-1}(G)]$.

The elements of the union are mutually disjoint and belong to $\mathscr{G}(\alpha)$. Hence, by Montgomery's theorem (see [1], p. 264-269), $h^{-1}(G) \in \mathscr{G}(\alpha)$.

Since G was arbitrary, the proof of the theorem is complete.

Proof of Theorem II. This part of the argument is due to Traczyk [3]. It is known that (cf. [1], p. 294, th. 3) whenever $\alpha > 0$ and $h \in \mathscr{B}(\alpha)$, there exists a sequence $\{h_n\}$ of functions such that $h_n \to h$ uniformly, the set $h_n(X)$ is isolated in Y, and $h_n \in \mathscr{B}(\alpha)$ for every n. (All sequences are of length ω from now on.)

Thus, by theorem 1, there is a sequence $\{f_n\}$ of functions, such that for each n, $f_n \in \mathscr{B}(\alpha)$, the set $f_n(X)$ is isolated in Y and

$$|f(x) - f_n(x)| < 2^{-n} \text{ (a. e.)}.$$

Hence the set

$$H_{n,1} = \{x : |f_n(x) - f_{n-1}(x)| < 3 \times 2^{-n}\}$$

belongs to $\mathscr{G}(\alpha)$. Using the fact that $f_n(X)$ and $f_{n-1}(X)$ are isolated, it is not difficult to prove that

$$X - H_{n,1} \in \mathscr{G}(\alpha).$$

Now define a double sequence of functions

$$\begin{array}{l} g_{1,1} \\ g_{2,1}, g_{2,2} \\ g_{3,1}, g_{3,2}, g_{3,3} \\ \cdots\cdots\cdots\cdots \end{array}$$

as follows: $g_{1,1} = f_1$ and for $n > 1$

$$g_{n,1} = \begin{cases} f_n(x) & \text{for} \quad x \in H_{n,1}, \\ f_{n-1}(x) & \text{for} \quad x \in X - H_{n,1}. \end{cases}$$

Suppose that the functions $g_{n,1}, g_{n,2}, \ldots, g_{n,m-1}$ for $n = 1, 2, \ldots$ are already defined. Let us write for $m > 2$

$$H_{n,m-1} = \{x : |g_{n,m-1}(x) - g_{n-1,m-1}(x)| < 3 \times 2^{-n}\}.$$

We put

$$g_{n,m}(x) = \begin{cases} g_{n,m-1}(x) & \text{for} \quad x \in H_{n,m-1}, \\ g_{n-1,m-1}(x) & \text{for} \quad x \in X - H_{n,m-1}. \end{cases}$$

Since, by the above,

$$|g_{n,m}(x) - g_{n-1,m-1}| < 3 \times 2^{-n}$$

for each x in X and $n, m = 2, 3, \ldots$ and since the space Y is complete, the sequence $\{g_{n,n}\}$ converges uniformly over X.

Since, for each n, $f_n \in \mathscr{B}(\alpha)$ and the set $f_n(X)$ is isolated in Y, the sets $H_{n,m}$ and $X - H_{n,m}$ belong to $\mathscr{G}(\alpha)$ and all the functions $g_{n,m}$ to $\mathscr{B}(\alpha)$.

Thus the function $g = \lim_{n\to\infty} g_{n,n}$ also belongs to $\mathscr{B}(\alpha)$. Let

$$Z = \bigcup_{n=1}^{\infty} \{x : |f(x) - f_n(x)| \geqslant 2^{-n}\}.$$

Then Z is a countable union of elements of $\mathcal{I}$. Thus, since $\aleph_0 \leqslant \aleph_\gamma$ and $\mathcal{I}$ is a σ_γ-ideal, $Z \in \mathcal{I}$. If $x \in X - Z$, then $|f_n(x) - f_{n-1}(x)| < 3 \times 2^{-n}$. Hence $x \in H_{n,1}$ and $g_{n,1}(x) = f_n(x)$.

By induction w. r. t. m, $g_{n,m}(x) = f_n(x)$ for any x in $X - Z$ and for each n and m provided $m \leqslant n$. Hence for any x in $X - Z$

$$g(x) = \lim_{n\to\infty} g_{n,n}(x) = \lim_{n\to\infty} f_n(x) = f(x).$$

We conclude that

$$g(x) = f(x) \text{ (a. e.)}.$$

This completes the proof of theorem II.

Proof of Theorem III. Assume the hypotheses of theorem III to hold. Since X is separable it has an enumerable base $\mathscr{R}$. Enumerate $\mathscr{R}$ as the sequence $\{R_n\}$. Denote by N the set of all positive integers n for which there is a function h_n in B such that

$$|f(x) - h_n(x)| < \varepsilon \text{ (a. e.) in } R_n.$$

As in the proof of theorem I, we can prove easily that N is infinite and that $\bigcup_{n \in N} R_n = X$.

Next, arrange N in a sequence $\{n_r\}$ and consider the sequence $\{h_{n_r}\}$ of functions. For each integer r there is a countable ordinal number η_r such that $h_{n_r} \in \mathscr{B}(\eta_r)$. Now the sequence $\{\eta_r\}$ of countable ordinal numbers has an upper bound α where $\alpha < \omega_1$. Hence, for any γ, $h_{n_r} \in \mathscr{B}(\alpha)$.

It is easily seen now that f has the property $\Delta(\alpha)$. Hence, by theorem II, there is a function g in $\mathscr{B}(\alpha)$ such that $g(x) = f(x)$ (a. e.). Since $\mathscr{B}(\alpha) \subset \mathscr{B}$, theorem III follows.

TWO COUNTER-EXAMPLES

1. If X is separable, then $\gamma = 0$ and the σ_γ-ideal $\mathcal{I}$ is a σ-ideal. The question arises whether the conclusions of theorems I and II still hold, if in case of a non-separable metric space X, the hypotheses of these theorems are modified to allow $\mathcal{I}$ to be a σ-ideal. The following example provides the negative answer.

Let X consist of points (x, y) of the closed unit square in the Euclidean plane with $(0, 0)$ as the lower left-hand vertex. Define the metric ϱ on X by

$$\varrho\{(x_1, y_1), (x_2, y_2)\} = \begin{cases} |x_2 - x_1| & \text{if} \quad y_1 = y_2, \\ 2 & \text{if} \quad y_1 \neq y_2. \end{cases}$$

The reader can verify that X is a non-separable metric space. Let R be the set of all points whose abscissae are rational and let f be the characteristic function of R. Further, let $\mathcal{I}$ be the class of all countable sets and put $a = 1$. The reader can verify that the modified conditions of theorems I and II are satisfied. Yet the conclusions of these theorems do not hold.

2. The question arises whether the conclusion of theorem III holds if the hypotheses of that theorem are modified to make X 1-separable and $\mathcal{I}$ a σ_1-ideal. The following example shows that it does not.

Let X be the Cartesian product of the closed unit interval and the set of all countable ordinals. Thus, the points of X are of the form (x, β) where $0 \leqslant x \leqslant 1$ and $0 \leqslant \beta < \omega_1$. Define the following metric ϱ on X:

$$\varrho\{(x_1, \beta_1), (x_2, \beta_2)\} = \begin{cases} |x_2 - x_1| & \text{if} \quad \beta_1 = \beta_2, \\ 2 & \text{if} \quad \beta_1 \neq \beta_2. \end{cases}$$

Let Y be the real line and let $\mathcal{I}$ consist of $\emptyset$ alone.

Now for each countable ordinal number η choose a real function f_η of Baire class η defined on $[0,1]$. Then, define f on X by

$$f(x, \beta) = f_\beta(x).$$

Since $\mathcal{I}$ consists of $\emptyset$ alone, (a. e.) means everywhere, and $\mathcal{I}$ is a σ_1-ideal. The reader will verify further that X is 1-separable, that f has the property Δ and that f is not a Borel-measurable function on X.

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TO WACŁAW SIERPIŃSKI
ON HIS 80-TH BIRTHDAY

ON SOME PROPERTIES OF HAMEL BASES

BY

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I dedicate this little note to Professor Wacław Sierpiński since I use in it methods which he used very successfully on so many occasions.

Throughout this paper $\alpha, \beta, \gamma, \ldots$ will denote ordinal numbers, $n_i, n_\alpha, \ldots$ integers, $r_\alpha, \ldots$ rational numbers, $r_\alpha^+, \ldots$ non-negative rationals and $a, a_\alpha, b, \ldots$ real numbers. H will denote a Hamel basis of the real numbers, H^* the set of all numbers of the form $\sum_\alpha n_\alpha a_\alpha$ $(a_\alpha \in H)$ (the sum is finite) and H^+ the set of all numbers of the form $\sum_\alpha r_\alpha^+ a_\alpha$ $(a_\alpha \in H)$. Measure will always be the Lebesgue measure, and (a, b) will denote the set of numbers $a < x < b$.

Sierpiński showed [1] that there are Hamel bases of measure 0 and also Hamel bases which are not measurable.

We are going to prove the following theorems:

THEOREM 1. *H^* is always non-measurable. In fact H^* has inner measure 0 and for every (a, b) the outer measure of $H^* \cap (a, b)$ is $b - a$.*

THEOREM 2. *Assume $c = \aleph_1$. Then there is an H for which H^+ has measure 0.*

Proof of Theorem 1. The sets $H^* + 1/n$, $2 \leqslant n < \infty$, are pairwise disjoint. Thus a simple argument shows that H^* has inner measure 0.

For every x there exists an n_x so that $n_x \cdot x$ is in H^*, or the sets $1/nH^*$, $2 \leqslant n < \infty$, cover the whole interval $(-\infty, +\infty)$. Hence H^* cannot have outer measure 0, and thus by the Lebesgue density theorem it has a point, say x_0, of outer density 1. But then (since H^* is an additive group) every point of $x_0 + H^*$ is a point of outer density 1 of H^*. Finally, it is easy to see that H^* is everywhere dense (since, if a and b are rationally independent, the numbers $n_1 a + n_2 b$ are everywhere dense).

Now it is easy to deduce that the outer measure of $H^* \cap (a, b)$ is $b - a$. To see this observe that since H^* has outer density 1 at x_0, for every $\varepsilon > 0$ there exist arbitrarily small values of η, such that the outer measure of $H^* \cap (x_0 - \eta, x_0 + \eta)$ is greater than $2(1 - \varepsilon)\eta$; but consequently the same holds for $H^* \cap (x_0 + t - \eta, x_0 + t + \eta)$, where t is an arbitrary