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ON THE REPRESENTATION OF THE CONTINUOUS FUNCTIONS OF TWO VARIABLES BY MEANS OF ADDITION AND CONTINUOUS FUNCTIONS OF ONE VARIABLE

BY

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A. N. Kolmogoroff [1] has proved the following important theorem:

For every $n \geqslant 2$ there exists functions $\psi^{pq}(x)$ real, increasing and continuous in the interval $E^1 = [0, 1]$, such that every function of n variables $f(x_1, \ldots, x_n)$ real and continuous in the cube E^n may be represented in the form:

$$f(x_1, \ldots, x_n) = \sum_{q=1}^{2n+1} \chi_q \Big[\sum_{p=1}^{n} \psi^{pq}(x_p)\Big],$$

where the functions $\chi_q(y)$ are real and continuous.

For $n = 2$ we get the representation

$$f(x_1, x_2) = \sum_{q=1}^{5} \chi_q[\psi^{1q}(x_1) + \psi^{2q}(x_2)]. \tag{1}$$

Our aim is to prove that the representation (1) above cannot be improved:

THEOREM. *The representation* (1) *of Kolmogoroff is best possible in the sense that if* $\psi^{pq}(x)$, $p = 1, 2$, $q = 1, \ldots, 4$, *are* 8 *fixed functions, real, increasing and continuous in* E^1, *then there exists a real and continuous function of* 2 *variables* $f(x_1, x_2)$ *which cannot be put in the form*

$$f(x_1, x_2) = \sum_{q=1}^{4} \chi_q[\psi^{1q}(x_1) + \psi^{2q}(x_2)], \tag{2}$$

where the $\chi_q(y)$ *are real and continuous.*

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(1) А. Н. Колмогоров, *О представлении непрерывных функций нескольких переменных в виде суперпозиций непрерывных функций одного переменного и сложения*, Доклады Академии Наук СССР 114 (1957), № 5, p. 953-956.

For the proof we use the following notation: $\alpha, \beta, \gamma, \delta$ being real numbers, the level curves

$$\psi^{11}(x_1)+\psi^{21}(x_2) = \alpha,$$
$$\psi^{12}(x_1)+\psi^{22}(x_2) = \beta,$$
$$\psi^{13}(x_1)+\psi^{23}(x_2) = \gamma,$$
$$\psi^{14}(x_1)+\psi^{24}(x_2) = \delta$$

will be denoted by $[\alpha]$, $[\beta]$, $[\gamma]$, $[\delta]$ respectively. A point common to two level curves, say $[\alpha]$, $[\delta]$ or $[\beta]$, $[\gamma]$, will be denoted $[\alpha\delta]$ or $[\beta\gamma]$ respectively. Also we put

$$\chi_1(\alpha) = A, \quad \chi_2(\beta) = B, \quad \chi_3(\gamma) = C, \quad \chi_4(\delta) = C,$$

and if a subscript or a superscript occurs in α, β, γ, δ it appears again in A, B, C, D, e. g. $\chi_3(\gamma_2') = C_2'$.

LEMMA 1. *Let $\psi^{pq}(x)$, $p = 1, 2$, $q = 1, 2$, be 4 real functions, increasing and continuous in the interval $E^1 = [0, 1]$. There exists a function $f(x_1, x_2)$ of 2 variables, continuous in E^2 and which cannot be written in the form*

$$f(x_1, x_2) = \sum_{q=1}^{2} \chi_q[\psi^{1q}(x_1)+\psi^{2q}(x_2)], \tag{3}$$

where the $\chi_q(y)$ are real.

Proof. Suppose we have obtained 4 distinct points of the form

$$[\alpha_1\beta_1], \quad [\alpha_1\beta_2], \quad [\alpha_2\beta_1], \quad [\alpha_2\beta_2]. \tag{4}$$

Consider a continuous function $f(x_1, x_2)$ equal to 1 at the points $[\alpha_1\beta_1]$, $[\alpha_2\beta_2]$ and equal to -1 at $[\alpha_1\beta_2]$, $[\alpha_2\beta_1]$. If f admitted the representation (3) we should have

$$A_1+B_1 = 1, \qquad A_2+B_2 = 1,$$
$$A_1+B_2 = -1 \qquad A_2+B_1 = -1.$$

These equations are contradictory, for by addition they give $2 = -2$. Hence f is not representable in the form (3).

To complete the proof we have to construct the 4 distinct points mentioned above.

We may suppose that two level curves $[\alpha]$, $[\beta]$ do not meet in two distinct points P_1, P_2, for, in this case, every function of the form (3) would have the same value at P_1 and P_2.

We conclude that a level curve $[\beta]$ cannot cover a square. Hence one of the increasing functions $\psi^{12}(x_1)$, $\psi^{22}(x_2)$ is strictly increasing and

therefore the coordinates of a point P of $[\beta]$ are continuous functions of a parameter t.

We deduce that *if $[\alpha_1]$, $[\alpha_2]$ meet $[\beta]$, then every level curve $[\alpha]$ for which α lies between α_1 and α_2 meets also $[\beta]$.* In fact, t_1, t_2 being the values of the parameter t for the points $[\alpha_1\beta]$, $[\alpha_2\beta]$ and putting

$$\psi^{11}(x_1(t))+\psi^{21}(x_2(t)) = \psi^1(t),$$

we have $\psi^1(t_1) = \alpha_1$, $\psi^1(t_2) = \alpha_2$. Hence, since ψ^1 is continuous: $\psi^1(t) = \alpha$, for some t lying between t_1 and t_2. This shows that $[\alpha]$ and $[\beta]$ intersect.

Let now $[\alpha_1]$ be any level curve. Choose 3 distinct points P_1, P_2, P_3 on $[\alpha_1]$ and let $[\beta_1]$, $[\beta_2]$, $[\beta_3]$ be the (distinct) 3 level curves through P_1, P_2, P_3. Let P_1', P_2', P_3' be 3 points situated on $[\beta_1]$, $[\beta_2]$, $[\beta_3]$ respectively but not on $[\alpha_1]$ and let $[\alpha_1']$, $[\alpha_2']$, $[\alpha_3']$ be the 3 level curves through these points. Two at least of the differences $\alpha_1-\alpha_1'$, $\alpha_1-\alpha_2'$, $\alpha_1-\alpha_3'$ will have the same sign, say $\alpha_1 < \alpha_1'$, $\alpha_1 < \alpha_2'$. Let α_2 be any number in the open intervals (α_1, α_1'), (α_1, α_2'). Since $[\alpha_1]$, $[\alpha_1']$ meet $[\beta_1]$ (in P_1 and P_1') we conclude that $[\alpha_2]$ meets $[\beta_1]$. In the same way $[\alpha_2]$ meets $[\beta_2]$. The 4 distinct points (4) are now constructed.

LEMMA 2. *Suppose that $\psi^{pq}(x)$, $p = 1, 2$, $q = 1, 2, 3$, are 6 fixed functions, real, increasing and continuous in E^1 and suppose that every function of 2 variables $f(x_1, x_2)$ continuous in E^2 may be written in the form:*

$$f(x_1, x_2) = \sum_{q=1}^{3} \chi_q[\psi^{1q}(x_1)+\psi^{2q}(x_2)], \tag{5}$$

where the $\chi_q(y)$ are continuous. γ being any number interior to the interval of variation of $\psi^{13}(x_1)+\psi^{23}(x_2)$ in E^2 and $\varepsilon > 0$ being given denote by $\Gamma_\gamma^\varepsilon$ the closed set of points (x_1, x_2) of E^2 for which $\gamma-\varepsilon \leqslant \psi^{13}(x_1)+\psi^{23}(x_2) \leqslant \leqslant \gamma+\varepsilon$. Then to every $\Gamma_\gamma^\varepsilon$ and every positive number k we can associate a continuous function $g(x_1, x_2)$ of modulus $\leqslant 1$, vanishing outside $\Gamma_\gamma^\varepsilon$, and 4 numbers $\beta, \beta', \beta'', \beta'''$ such that for every continuous function $f(x_1, x_2)$ coinciding with g in $\Gamma_\gamma^\varepsilon$ we have

$$\chi_2(\beta)-\chi_2(\beta')-\chi_2(\beta'')+\chi_2(\beta''') \geqslant k.$$

Proof. We shall give the proof for $k = 8$. For $k = 12, 16, 20, \ldots$ the proof would be the same with one or several more steps.

Let $[\gamma_1]$ be a level curve in $\Gamma_\gamma^\varepsilon$. *On $[\gamma_1]$ the function $\alpha_1 = \psi^{11}(x_1)+ +\psi^{21}(x_2)$ cannot remain constant.* In fact, suppose that $\psi^{11}(x_1)+\psi^{21}(x_2)$ is constant on $[\gamma_1]$. The function $\psi^{12}(x_1)+\psi^{22}(x_2)$ may not take the same value at 2 distinct points P_1, P_2 of $[\gamma_1]$, for, in such a case, we should have for any continuous function f of 2 variables: $f(P_1) = f(P_2)$. Thus the image of $[\gamma_1]$ by $\psi^{12}(x_1)+\psi^{22}(x_2)$ is an interval I not reduced to one point. Choose in E^2 a closed square S not meeting $[\gamma_1]$, but so small (and close to $[\gamma_1]$) that the value of $\psi^{12}(x_1)+\psi^{22}(x_2)$, for (x_1, x_2) in S, is in I. Let

$f(x_1, x_2)$ be a continuous function, vanishing in $[\gamma_1]$, but arbitrary in S. We may suppose, by adding a constant to each of the 2 functions χ_1, χ_3, that we have $\chi_1(\alpha_1) = \chi_3(\gamma_1) = 0$. We conclude that $\chi_2 = 0$ along the curve $[\gamma_1]$, i. e. that $\chi_2(\beta) = 0$ for any $\beta \in I$. In the square S we should have $f = \chi_1 + \chi_3$, which is impossible for the arbitrary function f, according to Lemma 1. This contradiction shows that $\psi^{11}(x_1)+\psi^{21}(x_2)$ is not constant on $[\gamma_1]$ and a similar argument shows that $\psi^{11}(x_1)+\psi^{21}(x_2)$ is not constant on any part of $[\gamma_1]$ situated in a square T no matter how small. In particular $[\gamma_1]$ or any other level curve cannot cover T.

Thus on $[\gamma_1]$ the function $\psi^{11}(x_1)+\psi^{21}(x_2)$ takes at least 9 distinct values, say $\alpha_1^1, \ldots, \alpha_1^9$. On each curve $[\alpha_1^i]$ choose a point P^i not situated on $[\gamma_1]$ and let $[\gamma^i]$ be the level curve through P^i. 5 at least of the differences $(\gamma_1 - \gamma^i)$ will have the same sign, say $\gamma_1 - \gamma^i > 0$ for $i = 1, \ldots, 5$. If γ_2 is any number situated in the intervals (γ_1, γ^i), $i = 1, \ldots, 5$, we see, as in the proof of Lemma 1 that $[\gamma_2]$ meets each of the curves $[\alpha_1^i]$, $i = 1, \ldots, 5$, at some point $[\alpha_1^i \gamma_2]$ and we can chose γ_2 as close as we like to γ_1.

Let $[\beta_2^i]$, $i = 1, \ldots, 5$, be the 5 level curves through the points $[\alpha_1^i \gamma_2]$. We can manage to obtain 5 distinct values β_2^i. To see this we show that one cannot have e. g. $\beta_2^1 = \beta_2^2$, except for one value of γ_2. In fact suppose that $\beta_2^1 = \beta_2^2$ and that for $\gamma_2' \neq \gamma_2$ the curves $[\beta_2^{1'}]$ and $[\beta_2^{2'}]$ through $[\alpha_1^1 \gamma_2']$ and $[\alpha_1^2 \gamma_2']$ coincide. The 4 points $[\alpha_1^1 \gamma_2]$, $[\alpha_1^2 \gamma_2]$, $[\alpha_1^1 \gamma_2']$, $[\alpha_1^2 \gamma_2']$ are distinct since $\alpha_1^1 \neq \alpha_1^2$ and $\gamma_2 \neq \gamma_2'$. If $f(x_1, x_2)$ is a continuous function equal to $+1, -1, -1, +1$ respectively at the mentioned points and if $f(x_1, x_2)$ admits the representation (5) we should have

$$A_1^1 + B_2^1 + C_2 = 1, \quad A_1^1 + B_2^{1'} + C_2' = -1,$$
$$A_1^2 + B_2^1 + C_2 = -1, \quad A_1^2 + B_2^{1'} + C_1' = 1,$$

whence the contradiction $0 = 4$.

Thus we can find a value γ_2 as close as we like to γ_1, and 5 distinct level curves $[\beta_2^i]$, $i = 1, \ldots, 5$, meeting $[\gamma_2]$ at the points $[\alpha_1^i \gamma_2]$, $i = 1, \ldots, 5$.

Next we can find a value γ_3 as close as we like to γ_2 and 3 distinct level curves $[\alpha_3^i]$, $i = 1, \ldots, 3$, meeting $[\gamma_3]$ at the points $[\beta_2^i \gamma_3]$, $i = 1, \ldots, 3$.

Finally we can find a value γ_4 as close as we like to γ_3 and 2 distinct level curves $[\beta_4^i]$, $i = 1, 2$, meeting $[\gamma_4]$ at the points $[\alpha_3^i \gamma_4]$, $i = 1, 2$.

Consider now the sequence of 8 points

$$\begin{array}{ll} [\alpha_1^1 \quad \gamma_1], & [\alpha_1^2 \quad \gamma_1], \\ [\alpha_1^1\ \beta_2^1\ \gamma_2], & [\alpha_1^2\ \beta_2^2\ \gamma_2], \\ [\alpha_3^1\ \beta_2^1\ \gamma_3], & [\alpha_3^2\ \beta_2^2\ \gamma_3], \\ [\alpha_3^1\ \beta_4^1\ \gamma_4], & [\alpha_3^2\ \beta_4^2\ \gamma_4]. \end{array}$$

Let $[\beta_0^1]$ and $[\beta_0^2]$ be the level curves through the first two of these points and let $g(x_1, x_2)$ be a continuous function of two variables, of modulus $\leqslant 1$, vanishing outside $\Gamma_\gamma^\varepsilon$, and taking the values $+1, -1, -1, +1, +1, -1, -1, +1$, respectively at the 8 points mentioned above.

If $f(x_1, x_2)$ is any function of 2 variables, continuous in E^2, coinciding with g in $\Gamma_\gamma^\varepsilon$, and admitting the representation (5), then

$$\begin{array}{ll} A_1^1 + B_0^1 + C_1 = 1, & A_1^2 + B_0^2 + C_1 = -1, \\ A_1^1 + B_2^1 + C_2 = -1, & A_1^2 + B_2^2 + C_2 = 1, \\ A_3^1 + B_2^1 + C_3 = 1, & A_3^2 + B_2^2 + C_3 = -1, \\ A_3^1 + B_4^1 + C_4 = -1, & A_3^2 + B_4^2 + C_4 = 1. \end{array}$$

Whence

$$B_0^1 - B_4^1 - B_0^2 + B_4^2 = 8,$$

i. e.

$$\chi_2(\beta_0^1) - \chi_2(\beta_4^1) - \chi_2(\beta_0^2) - \chi_2(\beta_4^2) = 8.$$

Lemma 2 is now proved.

LEMMA 3. *If $\psi^{pq}(x)$, $p = 1, 2$, $q = 1, 2, 3$, are 6 fixed functions, real, increasing and continuous in E^1, then there exists a real continuous function of 2 variables $f(x_1, x_2)$ which cannot be written in the form*

$$f(x_1, x_2) = \sum_{q=1}^{3} \chi_q[\psi^{1q}(x_1) + \psi^{2q}(x_2)], \tag{5}$$

where the $\chi_q(y)$ are continuous.

Proof. Let $\Gamma_{\gamma_n}^{\varepsilon_n}$ be a sequence of closed disjoint sets of the form indicated in the statement of Lemma 2. To every n we can associate a continuous function $g_n(x_1, x_2)$ of modulus $\leqslant 1/n^2$, vanishing outside $\Gamma_{\gamma_n}^{\varepsilon_n}$ and 4 numbers $\beta_n, \beta_n', \beta_n'', \beta_n'''$ such that, if $f(x_1, x_2)$ coincides with g_n in $\Gamma_{\gamma_n}^{\varepsilon_n}$ and if f admits the representation (5) then

$$\chi_2(\beta_n) - \chi_2(\beta_n') - \chi_2(\beta_n'') + \chi_2(\beta_n''') \geqslant n. \tag{6}$$

Put

$$f(x_1, x_2) = \sum_{n=1}^{\infty} g_n(x_1, x_2).$$

Then f is continuous in E^2. If f could be written in the form (5), relation (6) would be true for every n and the continuous function χ_2 would not be bounded. Thus f cannot be written in the form (5) with continuous $\chi_q(y)$.

COROLLARY. *Under the same conditions as in Lemma 3, if S is any square (closed or open) in E^2, then there exists a function $f(x_1, x_2)$ real and* *continuous in E^2 which does not admit the representation* (5) *in* S *with continuous* $\chi_q(y)$.

LEMMA 4. *Suppose that* $\psi^{pq}(x)$, $p=1,2$, $q=1,2,3,4$, *are* 8 *fixed functions real, increasing and continuous in* E^1, *and suppose that every function of* 2 *variables* $f(x_1,x_2)$, *continuous in* E^2 *may be written in the form*

$$f(x_1,x_2)=\sum_{q=1}^{4}\chi_q[\psi^{1q}(x_1)+\psi^{2q}(x_2)], \tag{2}$$

where the $\chi_q(y)$ *are continuous. To every open square* S_0 *in* E^2 *we can associate an open square* $S_1\subset S_0$ *such that if* $P_1=[\alpha_1\delta_1]$, $P_2=[\alpha_2\delta_2]$ *are any two points in* S_1, *then there are two points* Q_1, Q_2 *in* S_0 *of the form* $Q_1=[\alpha_2\delta_1]$, $Q_2=[\alpha_1\delta_2]$.

Proof. It is impossible that for every point $P=[\alpha\delta]$ of S_0 the two curves $[\alpha]$ and $[\delta]$ coincide in S_0. For, in such a case, we should have for some function $\varphi(y)$:

$$\psi^{14}(x_1)+\psi^{24}(x_2)=\varphi[\psi^{11}(x_1)+\psi^{21}(x_1)], \tag{7}$$

for every (x_1,x_2) in S_0. We see immediately that φ is continuous. But then, relations (2) and (7) show that an arbitrary continuous function of 2 variables would be representable in S_0 as sum of 3 functions χ_q. This is impossible by the Corollary to Lemma 3 and our assertion is proved.

Also, by the same Corollary, no level curve may cover a square, so that we conclude, as in the proof of Lemma 1, that the coordinates of a level curve are continuous functions of a parameter t.

Thus there exists a level curve $[\alpha_1']$ such that along this curve and inside S_0 the function $\delta=\psi^{14}(x_1)+\psi^{24}(x_2)$ takes at least 2 values (and even 3 values). We conclude as in the proof of Lemma 1 that there are 4 distinct points

$$P_1'=[\alpha_1'\delta_1'],\quad P_2'=[\alpha_2'\delta_2'],\quad Q_1'=[\alpha_2'\delta_1'],\quad Q_2'=[\alpha_1'\delta_2']$$

situated in S_0 (with $\alpha_1'\neq\alpha_2'$, $\delta_1'\neq\delta_2'$).

Observe now, since the $\psi^{pq}(x_p)$ are increasing, that every portion of a level curve limited by 2 points R_1, R_2 is entirely contained in the vertical rectangle of E^2 whose diagonal is R_1R_2.

Suppose, to fix the ideas, that $\alpha_1'<\alpha_2'$, $\delta_1'<\delta_2'$. Then, for every $\alpha\in(\alpha_1',\alpha_2')$, $\delta\in(\delta_1',\delta_2')$ there exists a point $P=[\alpha\delta]$ situated in S_0. In fact, since $[\alpha_1']$ and $[\alpha_2']$ meet $[\delta_1']$ in P_1' and Q_1', then the curve $[\alpha]$ meets $[\delta_2']$ at a point R_1 situated in the vertical rectangle whose diagonal is $P_1'Q_1'$. Similarly, $[\alpha]$ meets $[\delta_2']$ at a point R_2 situated in the vertical rectangle whose diagonal is P_2', Q_2'. We conclude that $[\delta]$ meets $[\alpha]$ at a point P situated in the vertical rectangle whose diagonal is R_1R_2 and therefore situated in S_0.

Choose now an open square $S_1\subset S_0$, containing P, and such that if $P_1=[\alpha_1\delta_1]\in S_1$, then $\alpha_1\in(\alpha_1',\alpha_2')$ and $\delta_1\in(\delta_1',\delta_2')$. If therefore P_1 and $P_2=[\alpha_2\delta_2]$ are two points of S_1, then there exists two points Q_1, Q_2 of the form $Q_1=[\alpha_2\delta_1]$, $Q_2=[\alpha_1\delta_2]$ situated in S_0. Lemma 4 is now proved.

LEMMA 5. *Let* $\mathfrak{A}$ *be an open square,* $2k$ *an even integer. We can find an open square* $\mathfrak{A}'\subset\mathfrak{A}$ *such that if* $[\alpha_1],\ldots,[\alpha_{2k}]$ *are level curves meeting* $\mathfrak{A}'$, *for which an* α *with an even subscript differs from an* α *with an odd subscript and if we start from a point* $[\alpha_1\gamma_0]$ *in* $\mathfrak{A}'$, *then the "$\beta\gamma$-alternating column of length* $2k$ *for the* α":

$$\begin{array}{l}[\alpha_1\ \beta_1\ \gamma_0]\\ [\alpha_2\ \beta_1\ \gamma_2]\\ [\alpha_3\ \beta_3\ \gamma_2]\\ [\alpha_4\ \beta_3\ \gamma_4]\\ \cdots\cdots\\ [\alpha_{2k}\ \beta_{2k-1}\ \gamma_{2k}]\end{array} \tag{8}$$

exists and the points are in $\mathfrak{A}$.

Proof. For simplicity we take $2k=4$, so that the last point is $[\alpha_4\beta_3\gamma_4]$.

Put $\mathfrak{A}_4=\mathfrak{A}$ and let $\mathfrak{A}_3\subset\mathfrak{A}_4$ be such that if $[\alpha_3\beta_3]$ and $[\alpha_4\beta_4]$ are in $\mathfrak{A}_3$, then a point $[\alpha_4\beta_3]$ exists in $\mathfrak{A}_4$. This comes to say that if $[\alpha_3\beta_3\gamma_2]$ is in $\mathfrak{A}_3$ and if $[\alpha_4]$ meets $\mathfrak{A}_3$ then $[\alpha_4\beta_3]$ is in $\mathfrak{A}_4$.

Next let $\mathfrak{A}_2\subset\mathfrak{A}_3$ be such that if $[\alpha_2\gamma_2]$ and $[\alpha_3\gamma_3]$ are in $\mathfrak{A}_2$, then a point $[\alpha_3\gamma_2]$ exists in $\mathfrak{A}_3$. This comes to say that if $[\alpha_2\beta_1\gamma_2]$ is in $\mathfrak{A}_2$ and if $[\alpha_3]$ meets $\mathfrak{A}_2$, then a point $[\alpha_3\gamma_2]$ exists in $\mathfrak{A}_3$.

Finally, take $\mathfrak{A}_1\subset\mathfrak{A}_2$ such that if $[\alpha_1\beta_1\gamma_0]$ is in $\mathfrak{A}_1$ and if $[\alpha_2]$ meets $\mathfrak{A}_1$, then $[\alpha_2\beta_1]$ exists in $\mathfrak{A}_2$.

Therefore, putting $\mathfrak{A}'=\mathfrak{A}_1$ and starting with a point $[\alpha_1\beta_1\gamma_0]$ in $\mathfrak{A}'$, we find successively that $[\alpha_2\beta_1\gamma_2]\in\mathfrak{A}_2$, $[\alpha_3\beta_3\gamma_2]\in\mathfrak{A}_3$, $[\alpha_4\beta_3]\in\mathfrak{A}_4=\mathfrak{A}$. Lemma 5 is now proved.

Definition. Instead of a $\beta\gamma$-alternating column of length $2k$ for the α, we define what may be called a *double $\beta\gamma$-alternating column of length k for the δ* as follows (where we suppose that all points of intersection considered do exist).

We start from 2 different level curves $[\gamma_0]$, $[\gamma_0']$ meeting $[\delta_0]$ and look for a $\delta_1>\delta_0$ such that the 2 points

$$P_1=[\beta_1\gamma_0\delta_1],\qquad P_1'=[\beta_1'\gamma_0'\delta_1]$$

have different β's. Then take a $\delta_2 > \delta_1$ such that the 2 points

$$P_2 = [\beta_1\gamma_2\delta_2], \quad P'_2 = [\beta'_1\gamma'_2\delta_2]$$

have different γ's. Next take a $\delta_3 > \delta_2$ such that the 2 points

$$P_3 = [\beta_3\gamma_2\delta_3], \quad P'_3 = [\beta'_3\gamma'_2\delta_3]$$

have different β's and then $\delta_4 > \delta_3$ such that the 2 points

$$P_4 = [\beta_3\gamma_4\delta_4), \quad P'_4 = [\beta'_3\gamma'_4\delta_4]$$

have different γ's, and so on.

If the process stops, i. e., if, for example, for every $\delta_5 > \delta_4$, in some neighbourhood of δ_4, the points

$$P_5 = [\beta_5\gamma_4\delta_5], \quad P'_5 = [\beta_5\gamma'_4\delta_5]$$

have the same β's, then take an increasing sequence $\delta_6, \ldots, \delta_k$, and put

$$P_6 = [\beta_6\gamma_4\delta_6], \quad P'_6 = [\beta_6\gamma'_4\delta_6],$$

$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots$$

$$P_k = [\beta_k\gamma_4\delta_k], \quad P'_k = [\beta_k\gamma'_4\delta_k].$$

Remark 1. We see easily, by using Lemmas 4 and 5 that we can always construct a double $\beta\gamma$-alternating column for the δ, for which all points are in a given open square $\mathfrak{D}$.

Remark 2. If a function $f(x_1\, x_2)$ admits the representation (2) and if it takes the values $+1, -1, \ldots, +1, -1$, at the points of a $\beta\gamma$-alternating column of length $2k$ for the α, then

$$(A_1-A_2+\ldots+A_{2k-1}-A_{2k})+(D_1-D_2+\ldots+D_{2k-1}-D_{2k})+C_0-C_{2k} = 2k.$$

While if f takes the values

$$f(P_1) = +1, \quad f(P'_1) = -1,$$
$$f(P_2) = -1, \quad f(P'_2) = +1,$$
$$f(P_3) = +1, \quad f(P'_3) = -1,$$
$$f(P_4) = -1, \quad f(P'_4) = +1,$$
$$\ldots\ldots \qquad \ldots\ldots$$

at the points of a double $\beta\gamma$-alternating column of length k for the δ, then

$$(A_1-A_2+\ldots\pm A_k)-(A'_1-A'_2+\ldots\pm A'_k)+$$
$$+4 \text{ terms at most in } B \text{ or } C = 2k.$$

In this case all the D's disappear.

Proof of the Theorem. Suppose that every continuous function $f(x_1, x_2)$ admits the representation (2).

Let S be an open square and k a given positive integer. We suppose, for the moment, that all points occuring in our construction exist and are in S, and that if a function is supposed to take the values $+1$ and -1 at 2 points, then these 2 points are different.

Consider a $\beta\gamma$-alternating column of length $2k$ for the α, say the column (8), and a function f taking values $+1, -1, \ldots, +1, -1$ at the points of this column. Then

$$\{1\} \quad (A_1-A_2+\ldots+A_{2k-1}-A_{2k})+(D_1-D_2+\ldots+D_{2k-1}-D_{2k})+$$
$$+2 \text{ terms at most in } C = 2k.$$

Some of the δ's in $\{1\}$ with an even subscript may be equal to some δ's with an odd subscript. In that case an even number of D's disappear from $\{1\}$ leaving only e. g.

$$\{1'\} \quad (A_1-A_2+\ldots+A_{2k-1}-A_{2k})+(D_{2k'+1}-D_{2k'+2}+\ldots+D_{2k-1}-D_{2k})+$$
$$+2 \text{ terms at most in } C = 2k.$$

But now a δ occuring in $\{1'\}$ with an even index is different from a δ with an odd index.

Consider a $\beta\gamma$-alternating column on length $2k-2k'$ for these remaining δ's and suppose that f takes the values $-1, +1, \ldots, -1, +1$ at the points of this column. Then

$$\{1''\} \quad -(D_{2k'+1}-D_{2k'+2}+\ldots+D_{2k-1}-D_{2k})-$$
$$-(A'_{2k'+1}-A'_{2k'+2}+\ldots+A'_{2k-1}-A'_{2k})+2 \text{ terms at most in } C = 2(k-k').$$

To this we add the points of a double $\beta\gamma$-alternating column of length k' for the δ and suppose that f takes the values $-1, +1, -1, \ldots$ and $+1, -1, +1, \ldots$ at the points of this double column; then, with a convenient notation

$$\{1'''\} \quad -(A'_1-A'_2+\ldots+A'_{2k'-1}-A'_{2k'})+$$
$$+4 \text{ terms at most in } B \text{ or } C = 2k'.$$

Then, by $\{1'\}, \{1''\}, \{1'''\}$:

$$\{2\} \quad (A_1-A_2+\ldots+A_{2k-1}-A_{2k})-(A'_1-A'_2+\ldots+A'_{2k-1}-A'_{2k})+$$
$$+2\cdot 6 \text{ terms at most in } B \text{ or } C = 2\cdot 2k.$$

There is no harm in writing "$2\cdot 6$ terms at most" instead of 8 terms.

Now we deal with the expression $(A_1'-A_2'+\ldots+A_{2k-1}'-A_{2k}')$ as we have dealt before with $(D_1-D_2+\ldots+D_{2k-1}-D_{2k})$, getting

$$\{3\}\quad (A_1-A_2+\ldots+A_{2k-1}-A_{2k})+(D_1'-D_2'+\ldots+D_{2k-1}'-D_{2k}')+$$
$$+3\cdot 6 \text{ terms at most in } B \text{ or } C = 3\cdot 2k.$$

After $2k$ steps in all we obtain

$$\{2k\}\quad (A_1-A_2+\ldots+A_{2k-1}-A_{2k})-(A_1^{(k)}-A_2^{(k)}+\ldots+A_{2k-1}^{(k)}-A_{2k}^{(k)})+$$
$$+2k\cdot 6 \text{ terms at most in } B \text{ or } C = 2k\cdot 2k.$$

This relation will be used later.

But we want first to justify our procedure.

The open square S being given, choose an open square $\mathfrak{D}_k$ whose closure is in S and let $\mathfrak{D}_k'\subset\mathfrak{D}_k$ be the square associated with $\mathfrak{D}_k$ by Lemma 5, so that if $[\delta_1^{(k)}],\ldots,[\delta_{2k}^{(k)}]$ is a sequence of level curves meeting $\mathfrak{D}_k'$, for which a $\delta^{(k)}$ with an even index is different from a $\delta^{(k)}$ with an odd index and if we start from a point $[\delta_1^{(k)}\gamma_0^{(k)}]$ in $\mathfrak{D}_k'$, then the $\beta\gamma$-alternating column of length $2k$ for the δ exists and the points are in $\mathfrak{D}_k$.

Take any curve $[\delta]$ meeting $\mathfrak{D}_k'$. We can find an interval I_δ such that if $\delta^{(k)}\in I_\delta^i$, then $[\delta^{(k)}]$ meets $\mathfrak{D}_k'$. The curve $[\delta]$ goes outside $\mathfrak{D}_k$. Take a point on $[\delta]$ outside $\mathfrak{D}_k$ and an open square $\mathfrak{A}_k$ in S, whose closure does not meet $\mathfrak{D}_k$, such that if $[\delta^{(k)}]$ meets $\mathfrak{A}_k$ then $\delta^{(k)}\in I_\delta$, and consequently $[\delta^{(k)}]$ meets $\mathfrak{D}_k'$. Then let $\mathfrak{A}_k'$ be the open square associated to $\mathfrak{A}_k$ by Lemma 5.

Now we operate with $\mathfrak{A}_k$ as we operated with $\mathfrak{D}_k$. We define an open square $\mathfrak{D}_{k-1}$ in S, whose closure does not meet $\mathfrak{A}_k\cup\mathfrak{D}_k$, and its associated square $\mathfrak{D}_{k-1}'$.

Next we define an open square $\mathfrak{A}_{k-1}$ in S, whose closure does not meet $\mathfrak{A}_k\cap\mathfrak{D}_k$ and its associated square $\mathfrak{A}_{k-1}'$.

The process can be continued until we reach an open square $\mathfrak{A}_1$ in S, not meeting $\mathfrak{D}_1\cup\mathfrak{A}_2\cup\mathfrak{D}_2\cup\ldots\cup\mathfrak{A}_k\cup\mathfrak{D}_k$, and its associated square $\mathfrak{A}_1'$.

Now we start with a $\beta\gamma$-alternating column of length $2k$ for the a say the column (8), for which the level curves $[a_1],\ldots,[a_{2k}]$ meet $\mathfrak{A}_1'$ and for which the a's with an even index are different from the a's with an odd index. We start from a point $[a_1\gamma_0]$ in $\mathfrak{A}_1'$. Then all the points of the $\beta\gamma$-alternating column are in $\mathfrak{A}_1$. The conditions imposed on f in $\mathfrak{A}_1$ can be realized.

Next the δ's occuring in column (8) are such that their level curves meet $\mathfrak{A}_1$ and consequently meet $\mathfrak{D}_1'$. Now if we consider a $\beta\gamma$-alternating column column of length $2(k-k')$ for some of these δ's, the corresponding points will be in $\mathfrak{D}_1$, and consequently are different from the points which occured earlier in $\mathfrak{A}_1$.

When $k'>0$ we have to add a double $\beta\gamma$-alternating column of length k' for the δ. We can always manage to have the corresponding points in $\mathfrak{D}_1$, with δ's that are different from the finite number of δ's which already occured in our construction.

Hence the conditions imposed on $f(x_1, x_2)$ in $\mathfrak{D}_1$ can be realized. And we can go on in this way.

Thus the procedure used to obtain equation $\{2k\}$ is justified.

This equation contains $4k+12k=16k$ terms at most on the l. h. s. while the r. h. s. is $4k^2$.

Take $k=n^3$. We may state the following result.

To every open square S_n we can associate a continuous function $g_n(x_1, x_2)$, equal to zero outside S_n, of modulus $\leqslant 1/n^2$, such that, for any continuous function $f(x_1, x_2)$ coinciding with g_n in S_n and admitting the representation (2) we have, for some values of a, β, γ, δ:

At most $16n^3$ *terms in* $A, B, C, D\geqslant(1/n^2)4n^6=4n^4$.

If we choose a sequence of disjoint S_n and if we put

$$f(x_1, x_2)=\sum_{n=1}^{\infty} g_n(x_1, x_2),$$

then $f(x_1, x_2)$ is continuous and coincides with g_n in S_n. Denote by M an upper bound for $|A|, |B|, |C|, |D|$. Then

$$16n^3M\geqslant 4n^4,\qquad M\geqslant n/4.$$

This last relation is impossible if the $\chi_a(y)$ are to be continuous. Therefore $f(x_1, x_2)$ cannot admit the representation (2) and the theorem is proved.

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