distinct from $a_0$ in common. Consequently for $n$ sufficiently large, the simple arc $h(L_{12})$ intersects one of the arcs $h(L_{1n}), h(L_{2n}), h(L_{3n})$, in a point $\neq a_0$. But this is impossible, because $h$ is a homeomorphism and for $n > 2$ the arcs $L_{1n}, L_{2n}, L_{3n}$, have with the arc $L_{12}$ only the point $a_0 = h(a_0)$ in common.

Thus the proof of the theorem is complete.

*REFERENCES*

[1] J. Charatonik, *On ramification points in the classical sense*, Fundamenta Mathematicae 51 (1962), p. 229-252.

[2] A. Lelek, *On plane dendroids and their end points in the classical sense*, ibidem 49 (1961), p. 301-319.

*Reçu par la Rédaction le 21. 5. 1962*

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COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 2

# *ON CANTORIAN MANIFOLDS IN A STRONGER SENSE*

BY

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Modifying the original definition of Cantorian manifolds, given by Urysohn in 1925, Alexandroff determined in 1957 (see [1] or § 1 below) a class of compacta that will be called here Cantorian manifolds in the stronger sense. The question has recently been raised by Borsuk whether every Cantorian manifold which is an ANR-set is a Cantorian manifold in the stronger sense. In the present note we answer this question in the affirmative for the 2-dimensional case (see § 3), and find a 3-dimensional counter-example (see § 4). Related topics are also examined.

**§ 1. Four kinds of Cantorian manifolds.** Roughly speaking, Cantorian manifolds are compacta whose separators have large dimensions. We recall that a set $S$ is said to be a *separator* of the space $X$ between the sets $A$ and $B$ if there exists a decomposition $X-S = M\cup N$ such that $\overline{M}\cap N = 0 = M\cap\overline{N}$, $A\subset M$ and $B\subset N$.

Let $X$ be a *compactum*, i. e. compact metric space. Following Alexandroff (see [1], p. 70), for every integer $n$, we consider the condition:

($\mathbf{U}^n$) *If $A, B\subset X$ are closed sets containing interior points, then every closed separator $S$ of $X$ between $A$ and $B$ satisfies*

$$n-1 \leqslant \dim S.$$

Evidently, condition ($\mathbf{U}^n$) is equivalent to the inequality $n\leqslant \operatorname{dc} X$ (see [5], p. 105). Since one always has $\operatorname{dc} X\leqslant \dim X$, and the Cantorian manifolds are characterized by the equality $\operatorname{dc} X = \dim X$ (ibidem), the following property (**U**) of the compactum $X$ is necessary and sufficient for $X$ to be a *Cantorian manifold*:

(**U**) *Condition* ($\mathbf{U}^n$) *holds for* $n = \dim X$.

Since, for compacta $S$, the inequality $n-1\leqslant \dim S$ is equivalent to the inequality $0 < d_{n-1}(S)$, where $d_m(S)$ denotes the $m$-dimensional degree of $S$ (see [5], p. 60), Alexandroff's modification of condition ($\mathbf{U}^n$) is the following (see [1], p. 70):

($\mathbf{V}^n$) *If $A, B\subset X$ are closed sets containing interior points, then there exists a number $\sigma > 0$ such that every closed separator $S$ of $X$ between*

*A and B satisfies*

$$\sigma < d_{n-1}(S).$$

Now, the compactum $X$ with the following property (**V**) will be called a *Cantorian manifold in the stronger sense*:

(**V**) *Condition* ($\mathbf{V}^n$) *holds for* $n = \dim X$.

A small change in the example given by Alexandroff (see [1], p. 68) yields a 2-dimensional locally connected Cantorian manifold which is not a Cantorian manifold in the stronger sense (see fig. 1). But it is not

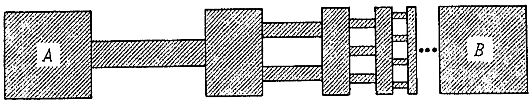


Fig. 1

an ANR-set. This gives a motivation to Borsuk's question mentioned at the beginning of the paper.

It is easily seen that each Cantorian manifold has the same dimension at each of its points. This leads to the definition of a third kind of Cantorian manifolds, namely of those which are distinguished among compacta $X$ by the following condition (see [1], p. 73):

($\mathbf{V}'$) *There is an integer* $n$ *such that* $\dim_x X = n$ *for* $x \in X$, *and if* $A, B \subset X$ *are closed sets satisfying*

$$\dim A \geqslant n \leqslant \dim B,$$

*then there exists a number* $\sigma > 0$ *such that every closed separator* $S$ *of* $X$ *between* $A$ *and* $B$ *satisfies*

$$\sigma < d_{n-1}(S).$$

Further, the uniformity of ($\mathbf{V}'$) can be defined as follows:

($\mathbf{V}'$)$_{\mathrm{u}}$ *There is an integer* $n$ *such that* $\dim_x X = n$ *for* $x \in X$, *and for each number* $\delta > 0$ *there exists a number* $\sigma > 0$ *such that if* $A, B \subset X$ *are closed sets satisfying*

$$d_n(A) > \delta < d_n(B),$$

*then every closed separator* $S$ *of* $X$ *between* $A$ *and* $B$ *satisfies*

$$\sigma < d_{n-1}(S).$$

At last, a fourth class of Cantorian manifolds is determined by a sharpening of condition ($\mathbf{V}'$) to the following one, which has been suggested by a result due to Sitnikov (compare [1], p. 74):



($\mathbf{V}''$) *There is an integer* $n$ *such that* $\dim_x X = n$ *for* $x \in X$, *and if* $A, B \subset X$ *are closed sets satisfying*

$$\dim A \geqslant m \leqslant \dim B,$$

*then there exists a number* $\sigma > 0$ *such that every closed separator* $S$ *of* $X$ *between* $A$ *and* $B$ *satisfies*

$$\sigma < d_{m-1}(S) \quad \text{or} \quad \sigma < d_m(S)$$

*for* $m = n$ *or* $m < n$, *respectively.*

Condition ($\mathbf{V}''$) implies Alexandroff's condition (**W**), which is formulated in a different manner (ibidem). The question whether these two conditions are equivalent for compacta in general, seems to be open.

As above in the case of condition ($\mathbf{V}'$), compacta for which condition ($\mathbf{V}''$) uniformly holds are the following:

($\mathbf{V}''$)$_{\mathrm{u}}$ *There is an integer* $n$ *such that* $\dim_x X = n$ *for* $x \in X$, *and for each number* $\delta > 0$ *there exists a number* $\sigma > 0$ *such that if* $A, B \subset X$ *are closed sets satisfying*

$$d_m(A) > \delta < d_m(B),$$

*then every closed separator* $S$ *of* $X$ *between* $A$ *and* $B$ *satisfies*

$$\sigma < d_{m-1}(S) \quad \text{or} \quad \sigma < d_m(S)$$

*for* $m = n$ *or* $m < n$, *respectively.*

Finally, let us remark that the compacta $X$ for which conditions ($\mathbf{V}^n$) or (**V**) uniformly hold (with respect to the diameters of massive spheres contained in $A$ and $B$, for instance) are actually the same for which conditions ($\mathbf{V}^n$) or (**V**) ordinarily hold, respectively. Thus neither the uniformity of ($\mathbf{V}^n$) nor that of (**V**) constitutes a new property.

**§ 2. Separators in locally connected compacta.** Let $A_1, A_2, \ldots$ be subsets of a compactum $X$. The symbols $\mathrm{Li}\,A_i$, $\mathrm{Ls}\,A_i$, and $\mathrm{Lim}\,A_i$, which we use in the sequel denote the topological limits of the sequence of sets $A_i$ in $X$ when the subscript index $i$ tends to the infinity (see [4], p. 241-245).

2.1. *Let* $X$ *be a locally connected compactum. If* $A_i, B_i \subset X$ *and* $S_i \subset X$ *is a separator of* $X$ *between* $A_i$ *and* $B_i$ *for* $i = 1, 2, \ldots$, *then* $\mathrm{Ls}\,S_i$ *is a separator of* $X$ *between* $\mathrm{Li}\,A_i - \mathrm{Ls}\,S_i$ *and* $\mathrm{Ls}\,B_i - \mathrm{Ls}\,S_i$.

Proof. Denote by $M$ the union of all the components of $X - \mathrm{Ls}\,S_i$ which intersect $\mathrm{Li}\,A_i$, and by $N$ the union of all the remaining components of $X - \mathrm{Ls}\,S_i$. Then $X - \mathrm{Ls}\,S_i = M \cup N$ and $\mathrm{Li}\,A_i - \mathrm{Ls}\,S_i \subset M$. Since every component of $X - \mathrm{Ls}\,S_i$ is an open set (see [4], p. 243 and [5], p. 163), $M$ and $N$ are disjoint open sets. Hence $\overline{M} \cap N = 0 = M \cap \overline{N}$.

To prove that $\mathrm{Ls}B_i - \mathrm{Ls}S_i \subset N$, suppose on the contrary that there is a point $q \in \mathrm{Ls}B_i - \mathrm{Ls}S_i$ such that $q \in X - N$. Thus there exists an infinite sequence of points $q_i$ satisfying $q_i \in B_{k_i}$ for $i = 1, 2, \ldots$, $\lim k_i = \infty$ and $\lim q_i = q$. Moreover, we have $q \in M$ and so we can find a component $C$ of $X - \mathrm{Ls}S_i$ which contains the point $q$ and a point $p \in \mathrm{Li}A_i$. Consequently, there exists an infinite sequence of points $p_i$ satisfying $p_i \in A_{k_i}$ for $i = 1, 2, \ldots$ and $\lim p_i = p$. The space $X$ being locally connected, let us take closed connected neighbourhoods $U$ and $V$ of points $p$ and $q$, respectively, such that $U \subset C$ and $V \subset C$. Since the set $C$ is connected and open in $X$, there is an arc $L \subset C$ which joins $p$ and $q$. The union $K = U \cup L \cup V$ is therefore a continuum, which contains both points $p_i$ and $q_i$ for a sufficiently large $i$. It follows that the set $S_{k_i}$ meets the continuum $K$ for a sufficiently large $i$, and we conclude by the equality $\lim k_i = \infty$ that the sets $\mathrm{Ls}S_i$ and $K$ intersect, which is impossible because $K \subset C \subset X - \mathrm{Ls}S_i$. This completes the proof of 2.1.

It is clear that the locally connected continuum considered in § 1 (see fig. 1) admits irreducible closed separators with arbitrarily large number of components. However, this continuum is not $\mathrm{LC}^1$, i. e. locally connected in dimensions 0 and 1 (see [5], p. 506).

2.2. *If $X$ is an $\mathrm{LC}^1$ compactum, then there exists an integer $m$ such that every irreducible closed separator of $X$ between two points consists of at most $m$ components.*

Proof. Using Eilenberg's notation, put $m = r(X)+1$ (see [2], p. 153). Since $X$ is $\mathrm{LC}^1$, $r(X)$ is finite (see [2], p. 175 and [3], p. 117). If $S$ is an irreducible closed separator of $X$ between two points, then there is a component $C$ of $X - S$ such that $\bar{C} - C = S$ (see [5], p. 175). Hence $X = \bar{C} \cup (X - C)$ is a decomposition of $X$ into continua (ibidem, p. 88, 163 and 175) whose common part is $S$. This shows that the number $m$ satisfies 2.2.

**§ 3. Existence of finite separators.** If a space is cut by a finite set $F$, the cutting $F$ fulfils the condition $d_1(F) < \varepsilon$ for every $\varepsilon > 0$. With the restriction to $\mathrm{LC}^1$ compacta the inverse holds too.

3.1. *Let $X$ be an $\mathrm{LC}^1$ compactum. If $A_i, B_i \subset X$ are connected sets and $S_i \subset X$ is a separator of $X$ between $A_i$ and $B_i$ satisfying*

$$(*) \qquad d_1(S_i) < 1/i \quad \text{for} \quad i = 1, 2, \ldots,$$

*then there exists an infinite sequence of integers $k_1 < k_2 < \ldots$ and a finite set $F$ such that*

$$F \subset \mathrm{Ls}S_{k_i}$$

*and $F$ is a separator of $X$ between $\mathrm{Li}A_{k_i} - F$ and $\mathrm{Ls}B_{k_i} - F$.*

Proof. The space $X$ being locally connected, the separator $S_i$ contains an irreducible closed separator $I_i$ of $X$ between $A_i$ and $B_i$ for $i = 1, 2, \ldots$ (see [5], p. 97 and 176). Let $k_1 < k_2 < \ldots$ be such a sequence of integers that the sequence $I_{k_1}, I_{k_2}, \ldots$ is convergent, and put $F = \mathrm{Lim}I_{k_i}$ (ibidem, p. 21). Thus $F \subset \mathrm{Ls}S_{k_i}$. Since $d_1(I_{k_i}) \leqslant d_1(S_{k_i}) < 1/k_i$ for $i = 1, 2, \ldots$, the diameters of components of $I_{k_i}$ converge to zero when $i$ tends to the infinity. It follows from 2.2 that the set $F$ is finite. The last assertion from 3.1 is a consequence of 2.1 because $F = \mathrm{Ls}I_{k_i}$.



3.2. *Let $X$ be an $\mathrm{LC}^1$ compactum. If $A, B \subset X$ are connected sets and $S_i \subset X$ is a separator of $X$ between $A$ and $B$ satisfying $(*)$, then there exists a finite separator $F \subset \mathrm{Ls}S_i$ of $X$ between $A - F$ and $B - F$.*

Proof. Putting $A_i = A$ and $B_i = B$ for $i = 1, 2, \ldots$, it is sufficient to apply 3.1.

Note that conditions $(\mathbf{U}^0)$ and $(\mathbf{V}^0)$ (see § 1) can be understood to hold for each space $X$. Similarly, conditions $(\mathbf{U}^1)$ and $(\mathbf{V}^1)$ are always equivalent. Of course, conditions $(\mathbf{U}^2)$ and $(\mathbf{V}^2)$ are not equivalent in general (see fig. 1).

3.3. *If $X$ is an $\mathrm{LC}^1$ compactum, then conditions $(\mathbf{U}^2)$ and $(\mathbf{V}^2)$ are equivalent.*

Proof. Clearly, $(\mathbf{V}^2)$ implies $(\mathbf{U}^2)$. Suppose $(\mathbf{V}^2)$ is not true. Then there are such closed subsets $A, B \subset X$ containing interior points that there exists, for every $i = 1, 2, \ldots$, a closed separator $S_i$ of $X$ between $A$ and $B$ satisfying $(*)$. If at least one of the sets $A$ and $B$ contains only degenerate continua, a massive sphere contained in it is 0-dimensional (see [5], p. 130), whence the space $X$ is not connected, and so $(\mathbf{U}^2)$ is not true. If both sets $A$ and $B$ contain non-degenerate continua, say $A'$ and $B'$, respectively, it follows from 3.2 that there is a finite separator $F$ of $X$ between $A' - F$ and $B' - F$, and there are points $p \in A' - F$ and $q \in B' - F$. The 0-dimensional set $F$ is thus a separator of $X$ between sufficiently small closed neighbourhoods of $p$ and $q$, which means that $(\mathbf{U}^2)$ is not true, and 3.3 is proved.

3.4. *If a 2-dimensional Cantorian manifold is an ANR-set (or only $\mathrm{LC}^1$), it is a Cantorian manifold in the stronger sense.*

This directly follows from 3.3.

3.5. *If $X$ is an $\mathrm{LC}^1$ compactum and $\dim X \leqslant 2$, then all conditions $(\mathbf{U})$, $(\mathbf{V})$, $(\mathbf{V}')$, $(\mathbf{V}')_{\mathrm{u}}$, $(\mathbf{V}'')$ and $(\mathbf{V}'')_{\mathrm{u}}$ are equivalent.*

Proof. Evidently, condition $(\mathbf{U})$ is implied by and condition $(\mathbf{V}'')_{\mathrm{u}}$ implies each of the others. It is enough to show that $(\mathbf{U})$ implies $(\mathbf{V}'')_{\mathrm{u}}$.

Suppose $(\mathbf{V}'')_{\mathrm{u}}$ does not hold and denote $n = \dim X$. If $\dim_x X < n$ for any $x \in X$, condition $(\mathbf{U})$ does not hold. We can thus assume that $\dim_x X = n$ for every $x \in X$. Consequently, there exist a number $\delta > 0$, an integer $m \leqslant n$, closed sets $A_i, B_i \subset X$, and closed separators $S_i$ of $X$

between $A_i$ and $B_i$ $(i = 1, 2, \ldots)$ such that

$$(**) \qquad d_m(A_i) > \delta < d_m(B_i),$$

$$(***) \qquad \frac{1}{i} > \begin{cases} d_{m-1}(S_i) & \text{if} \quad m = n, \\ d_m(S_i) & \text{if} \quad m < n, \end{cases}$$

for $i = 1, 2, \ldots$ Then $n = 1$ or 2. If $n = 1$, the degree $d_0(S_1)$ is finite by (***), and so it is zero, which means that the separator $S_1$ is empty while neither the set $A_1$ nor $B_1$ is so, according to (**). Therefore (**U**) does not hold. If $n = 2$ and $m = 0$, (**U**) does not hold for the same reason.

If $n = 2$ and $m > 0$, we have (**) for $m = 1$, the inequality $d_2(Y) \leqslant d_1(Y)$ being always true. Since $X$ is a compactum, we can assume that the sequences $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ are both convergent (ibidem, p. 21) and all their terms are continua (ibidem, p. 64 and 122). Put $A = \operatorname{Lim} A_i$ and $B = \operatorname{Lim} B_i$. Inequalities (**) for $m = 1$ imply the inequalities $d_1(A) \geqslant \delta \leqslant d_1(B)$ (ibidem, p. 61), whence $A$ and $B$ are infinite sets. Further, (***) yields (*) and we infer from 3.1 that there exists an infinite sequence of integers $k_1 < k_2 < \ldots$ and a finite separator $F$ of $X$ between $\operatorname{Li} A_{k_i} - F$ and $\operatorname{Ls} B_{k_i} - F$. But since $A \subset \operatorname{Li} A_{k_i}$ (see [4], p. 242 and 245) and $B \subset \operatorname{Li} B_{k_i} \subset \operatorname{Ls} B_{k_i}$ (ibidem, p. 244), the sets $\operatorname{Li} A_{k_i} - F$ and $\operatorname{Ls} B_{k_i} - F$ are not empty. Taking points $p$ and $q$ in these sets, respectively, we see that $F$ is a 0-dimensional separator of $X$ between sufficiently small closed neighbourhoods of $p$ and $q$. As $\dim X = n = 2$, (**U**) does not hold, and the proof of 3.5 is completed.

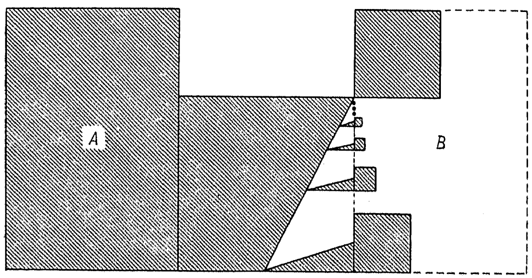


Fig. 2

3.6. *Let $X$ be an* $LC^1$ *compactum. If $A, B, C \subset X$ are sets such that $A$ and $B$ consist of a finite number of components and, for every number $\varepsilon > 0$ and every open set $G \subset X$ containing $C$, there is a separator $S \subset G$ of $X$ between $A$ and $B$ satisfying $d_1(S) < \varepsilon$, then $\bar{C}$ contains a finite separator $F$ of $X$ between $A - F$ and $B - F$.*



Proof. The theorem is readily reduced to the case where $A$ and $B$ are connected sets. According to the hypotheses concerning $C$, we can choose an infinite sequence of separators $S_1, S_2, \ldots$ of $X$ such that (*) is satisfied and $S_i$ lies in the $(1/i)$-neighbourhood of $C$ for $i = 1, 2, \ldots$ Then $\operatorname{Ls} S_i \subset \bar{C}$ and 3.6 follows from 3.2.

Let us observe that theorem 3.6 fails for $A$ or $B$ consisting of an infinite number of components (see fig. 2).

In the above example the sets $A, B$, and $C$ are closed; however the set $C$, which consists of all cut points, intersects the set $B$. As the next theorem states, under the condition that all the sets $A, B$, and $C$ are closed and the set $C$ does not meet the union $A \cup B$, the hypothesis in 3.6 that $A$ and $B$ consist of finitely many components can be omitted.

3.7. *Let $X$ be an* $LC^1$ *compactum. If $A, B, C \subset X$ are closed sets such that $A \cup B \subset X - C$ and, for every number $\varepsilon > 0$ and every open set $G \subset X$ containing $C$, there is a separator $S \subset G$ of $X$ between $A$ and $B$ satisfying $d_1(S) < \varepsilon$, then $C$ contains a finite separator of $X$ between $A$ and $B$.*

Proof. First we prove that the compactum $A \cup B$ is contained in the union of finitely many components of $X - C$. Indeed, suppose on the contrary that there is an infinite sequence of components $G_1, G_2, \ldots$ of $X - C$ which intersect $A \cup B$. Then they are open sets (see [5], p. 163) and have boundaries $\operatorname{Fr}(G_i) \subset \operatorname{Fr}(X - C) \subset C$ for $i = 1, 2, \ldots$ (ibidem, p. 169). Consequently, $\operatorname{Ls} G_i \subset \operatorname{Ls} \operatorname{Fr}(G_i) \subset C$ (ibidem), which gives

$$0 \neq (A \cup B) \cap \operatorname{Ls} G_i \subset (A \cup B) \cap C,$$

contrary to the hypothesis.

Thus there are components $G_1, \ldots, G_k$ of $X - C$ such that $A \subset G_1 \cup \ldots \cup G_k$ and $A \cap G_i \neq 0$ for $i = 1, \ldots, k$. Since $A \cap (\bar{G}_i - G_i) \subset A \cap C = 0$, the set $A \cap G_i$ is compact for $i = 1, \ldots, k$. There exist now continua $K_i \subset G_i$ such that $A \cap G_i \subset K_i$ for $i = 1, \ldots, k$ (ibidem, p. 167). Hence

$$A \subset K = K_1 \cup \ldots \cup K_k \subset X - C, \qquad A \cap K_i \neq 0$$

for $i = 1, \ldots, k$. Similarly, there exist continua $L_i$ $(i = 1, \ldots, l)$ such that

$$B \subset L = L_1 \cup \ldots \cup L_l \subset X - C, \qquad B \cap L_i \neq 0$$

for $i = 1, \ldots, l$. It follows that every closed separator of $X$ between $A$ and $B$ which does not meet the union $K \cup L$ is a separator of $X$ between $K$ and $L$. The set $X - (K \cup L)$ being open and containing $C$, the hypotheses of 3.7 hold for $K$ and $L$ instead of $A$ and $B$, respectively. But the compacta $K$ and $L$ consist of at most $k$ and $l$ components, respectively. Applying 3.6, we obtain a finite separator $F \subset \bar{C} = C$ of $X$ between $K - F$ and $L - F$. Since $A \subset K = K - F$ and $B \subset L = L - F$, the proof of 3.7 is concluded.

3.8. *Let $X$ be an* $LC^1$ *compactum. If $A, B \subset X$ are closed sets and $C$ is a 0-dimensional separator of $X$ between $A$ and $B$, then $C$ contains a finite separator of $X$ between $A$ and $B$.*

This instantly follows from 3.7, if one observes that $C$ in 3.8 may be assumed to be closed (ibidem, p. 97) and then it has already all the properties of $S$ required in 3.7.

3.9. *Let $X$ be an* $LC^1$ *compactum. If $A, B \subset X$ are closed sets and $C$ is a separator of $X$ between $A$ and $B$ such that $\dim_x X \leqslant 1$ for $x \in C$, then $C$ contains a finite separator of $X$ between $A$ and $B$.*

Proof. As previously, we can assume that $C$ is closed. Take an arbitrary open set $G \subset X$ containing $C$. Since $C$ is a separator of $X$ between $A$ and $B$, we have $C \subset G-(A \cup B)$. Choose a finite cover $V_1, \ldots, V_j$ of $C$ composed of open sets $V_i$ with 0-dimensional (or empty) boundaries and closures $\overline{V}_i$ contained in $G-(A \cup B)$. This exists according to the hypothesis on $C$. Let $S$ be the union of boundaries of the sets $V_1, \ldots, V_j$. $C$ being a separator of $X$ between $A$ and $B$, so is $S$. Moreover, $\dim S \leqslant 0$, whence $d_1(S) = 0$, and 3.9 follows from 3.7.

3.10. *If $X$ is a 1-dimensional ANR-set, then every separator of $X$ between closed sets contains a finite separator of $X$ between these sets.*

This is an immediate consequence of 3.9, as 1-dimensional $LC^1$ compacta and 1-dimensional ANR-sets coincide (see [5], p. 289).

Remark that if $X$ is a 1-dimensional ANR-set, $A$ and $B$ range over all the subcontinua of $X$, and $C$ ranges over all the separators of $X$ between $A$ and $B$, then the minimal number $m(X; A, B, C)$ of points of $C$ constituting the finite separator of $X$ that exists (by 3.10) between $A$ and $B$ is bounded (by 2.2), i. e. one can find an integer $m(X)$ satisfying $m(X; A, B, C) \leqslant m(X)$ for all continua $A, B$ and separators $C$. Simple examples show that the connectedness of $A$ and $B$ is necessary here.

**§ 4. Some 3-dimensional AR-sets.** The first of the continua which are described in this paragraph is the counter-example announced at the beginning of the paper.

4.1. *There exists a 3-dimensional AR-set that satisfies condition* (U) *but not* (V).

Proof. Let $P_1$ and $P_2$ be 3-cells in the 3-dimensional Euclidean space whose common part is a straight segment $I$ (see fig. 3). Take an arbitrary continuous mapping $g$ of $I$ onto the square $I^2$ and consider the upper semicontinuous decomposition of the union $P = P_1 \cup P_2$ into the sets $g^{-1}(y)$, where $y \in I^2$, and the points belonging to $P-I$. According to the well known Alexandroff theorem (compare [5], p. 42), this decomposition induces a continuous mapping $f$ of $P$ such that $f|P-I$



is a homeomorphism of $P-I$ onto $f(P)-f(I)$ and the set $f(I)$ is homeomorphic to the set $g(I) = I^2$, i. e. $f(I)$ is a 2-cell. Since $P, I$, and $f(I)$ are ANR-sets, the image $X = f(P)$ is also an ANR-set by virtue of the Borsuk theorem (ibidem, p. 264). The ANR-set $X$ being obviously contractible, it is an AR-set (ibidem, p. 289).

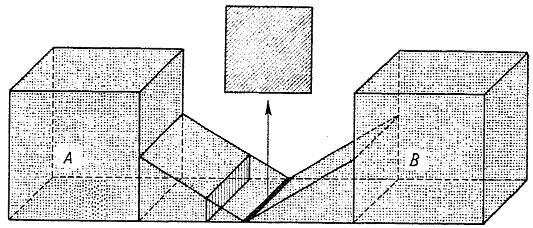

Fig. 3

To prove (U), it is enough to show that no closed subset $Y \subset X$ with dimension $\dim Y \leqslant 1$ is a separator of $X$. In fact, let $R_i$ denote the set of interior points of $P_i$ $(i = 1, 2)$. Since $f|R_i$ is a homeomorphism, we have $\dim f^{-1}(Y) \cap R_i \leqslant 1$ and we infer from the Mazurkiewicz theorem (ibidem, p. 343) that $R_i - f^{-1}(Y)$ is a connected set, dense in $P_i$, for $i = 1$ and 2. Consequently, its image under $f$ is a connected dense subset of $f(P_i)$. But the set $f(I)-Y$ is not empty and lies in the common part of $f(P_1)$ and $f(P_2)$. It follows that the union

$$f[R_1 - f^{-1}(Y)] \cup [f(I)-Y] \cup f[R_2 - f^{-1}(Y)],$$

contained in $X-Y$, is a connected dense subset of $X$. Thus $X-Y$ is a connected set.

Now, let $A$ and $B$ be closed subsets of $P$, with interior points, contained in $P_1 - I$ and $P_2 - I$, respectively (see fig. 3). By the continuity of $f$, for every number $\sigma > 0$ there exists a number $\varepsilon > 0$ such that $\varrho(p, p') < \varepsilon$ implies $\varrho[f(p), f(p')] < \sigma/2$ for $p, p' \in P$. A rectangle $T$, sufficiently narrow, near but disjoint with $I$ (see fig. 3), cuts $P$ between $A$ and $B$, and satisfies the inequality $d_2(T) < \varepsilon$. Then $f(T)$ is a separator of $X$ between $f(A)$ and $f(B)$, and we have $d_2[f(T)] < \sigma$, the mapping $f|T$ being a homeomorphism. Since $\dim X = 3$, condition (V) does not hold for the AR-set $X$, and 4.1 is proved.

4.2. *There exists a 3-dimensional AR-set that satisfies condition* (V) *but not* (V′).

Proof. Let $P'$ be a 3-cell and $I'$ an arc on the boundary of $P'$. Take a mapping $g'$ of $I'$ onto $P'$. Quite similarly as in the preceding construction (with $P$, $I$ and $g$ replaced by $P'$, $I'$ and $g'$, respectively), $g'$ yields a mapping $f'$ of $P'$ such that $X' = f'(P')$ is a 3-dimensional AR-set. Every open subset of $X'$ contains a 3-cell which is the image under $f'$ of a 3-cell contained in the interior of $P'$. Consequently, condition (**V**) holds for the continuum $X'$ because of the fact that (**V**) is satisfied by each 3-cell (see [1], p. 71). However, condition (**V**$'$) does not hold for $X'$, if one chooses $A' = f'(I')$ and $B'$ equal to an arbitrary 3-cell lying in $f'(P'-I')$.

4.3. *There exists a 3-dimensional* AR-*set that satisfies condition* $(\mathbf{V}')_u$ *but not* (**V**$''$).

Proof. The required AR-set $X''$ is a part of the AR-set $X$ constructed in 4.1, namely $X'' = f(P_1)$. It can be verified that for each number $\delta > 0$ there exist a number $\eta > 0$ and a 3-cell $Q$, contained in $f(P_1 - I)$, with the property: if $Z \subset X''$ is a closed subset satisfying the inequality $\delta < d_3(Z)$, then $\eta < d_3(Q \frown Z)$. We infer that condition $(\mathbf{V}')_u$ holds for $X''$, since it holds for each 3-cell (ibidem). Further, taking $A'' = f(I)$ and $B''$ equal to an arbitrary 2-cell contained in $f(P_1 - I)$, we can find for every number $\sigma > 0$ a sufficiently narrow and near $I$ rectangle $T \subset P_1 - I$ such that its image $S = f(T)$ is a separator of $X''$ between $A''$ and $B''$, and we have $d_2(S) < \sigma$. Consequently, condition (**V**$''$) fails for $X''$ (with $m = 2$), and so proposition 4.3 is shown.

Observe that the above AR-set $X''$ in fact does not satisfy even Alexandroff's condition (**W**), weaker than condition (**V**$''$) at first sight (see § 1). Really, the separator $S$ with an arbitrarily small 2-dimensional degree can be found in an arbitrarily tight neighbourhood of $A''$.

**§ 5. Final remarks.** Denoting implications by arrows, we can write down the obvious relations between classes of Cantorian manifolds in the following diagram:

$$\begin{array}{ccccccc} (\mathbf{U}) & \leftarrow & (\mathbf{V}) & \leftarrow & (\mathbf{V}') & \leftarrow & (\mathbf{V}'') \\ & & & & \uparrow & & \uparrow \\ & & & & (\mathbf{V}')_u & \leftarrow & (\mathbf{V}'')_u \end{array}$$

By 3.5, all these implications become equivalences for 2-dimensional $LC^1$ compacta. There are easy examples (similar to the example given in § 1, fig. 1) of 2-dimensional locally connected continua, thus $LC^0$ compacta, for which we should reverse none of the above arrows, respectively.

According to 4.1, 4.2 and 4.3, no horizontal arrow would be reversed for 3-dimensional ANR-sets. The question concerning vertical arrows remains open:

**P 419.** *Do* (**V**$'$) *and* (**V**$''$) *imply* $(\mathbf{V}')_u$ *and* $(\mathbf{V}'')_u$, *respectively, for* ANR-*sets with dimensions* $n = 3, 4, \ldots$?



### REFERENCES


[1] P. Alexandroff, *Die Kontinua* ($\mathbf{V}^p$) *— eine Verschärfung der Cantorschen Mannigfaltigkeiten*, Monatshefte für Mathematik 61 (1957), p. 67-76.

[2] S. Eilenberg, *Sur les espaces multicohérents I*, Fundamenta Mathematicae 27 (1936), p. 153-190.

[3] — *Sur les espaces multicohérents II*, ibidem 29 (1937), p. 101-122.

[4] C. Kuratowski, *Topologie I*, Warszawa 1958.

[5] — *Topologie II*, Warszawa 1961.


*Reçu par la Rédaction le 15. 5. 1962*