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A COUNTABLE BROOM WHICH CANNOT BE IMBEDDED IN THE PLANE

BY

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By a *broom* we mean here a curve B (i. e. 1-dimensional continuum) which is a union of simple arcs L_μ, $\mu \in M$, with only one common point a (vertex of the broom). If the set M of indices μ is countable, i. e.

$$B = \bigcup_{n=1}^{\infty} L_n, \quad L_m \cap L_n = \{a\} \quad \text{for } m \neq n,$$

then B is said to be a *countable broom*. Countable brooms constitute a subclass of the class of *dendroids*, i. e. arcwise connected and hereditarily acyclic curves, recently investigated by J. Charatonik [1] and A. Lelek [2].

The aim of the present note is to prove the following

THEOREM. *There exists a countable broom B which cannot be topologically imbedded in the Euclidean plane E^2.*

Proof. Let (x, y, z) denote the point of the Euclidean 3-space E^3 with the Cartesian coordinates x, y and z. Let

$$a_0 = (0, 0, 0), \quad a_1 = (1, 0, 0), \quad a_2 = (0, 1, 0), \quad a_3 = (0, -1, 0).$$

For every $n = 2, 3, \ldots$ let us set

$$a_{0n}^- = (0, 0, -1/n), \qquad a_{1n}^- = (1+1/n, 0, -1/n),$$
$$a_{1n}^+ = (1+1/n, 0, 1/n), \qquad a_{2n}^- = (0, 1+1/n, -1/n),$$
$$b_{0n}^- = (0, 0, -\sqrt{2}/n), \qquad b_{1n}^- = (1+\sqrt{2}/n, 0, -\sqrt{2}/n),$$
$$b_{1n}^+ = (1+\sqrt{2}/n, 0, \sqrt{2}/n), \qquad b_{3n}^+ = (0, -1-\sqrt{2}/n, -\sqrt{2}/n),$$
$$c_{2n}^- = (0, 1+\sqrt{3}/n, -\sqrt{3}/n), \qquad c_{2n}^+ = (0, 1+\sqrt{3}/n, \sqrt{3}/n),$$
$$c_{3n}^- = (0, -1-\sqrt{3}/n, -\sqrt{3}/n).$$

Let us denote by $\overline{pq}$ the segment in E^3 with endpoints $p, q \in E^3$, and let us set

$$L_{01} = \overline{a_0 a_1}, \quad L_{02} = \overline{a_0 a_2} \quad L_{03} = \overline{a_0 a_3}$$

and for every $n = 2, 3, \ldots$

$$L_{1n} = \overline{a_0 a_{1n}^+} \cup \overline{a_{1n}^+ a_{1n}^-} \cup \overline{a_{1n}^- a_{0n}^-} \cup \overline{a_{0n}^- a_{2n}^-},$$

$$L_{2n} = \overline{a_0 b_{1n}^+} \cup \overline{b_{1n}^+ b_{1n}^-} \cup \overline{b_{1n}^- b_{0n}^-} \cup \overline{b_{0n}^- b_{3n}^-},$$

$$L_{3n} = \overline{a_0 c_{2n}^+} \cup \overline{c_{2n}^+ c_{2n}^-} \cup \overline{c_{2n}^- c_{3n}^-}.$$

It is easily seen that $L_{01}, L_{02}, L_{03}, L_{1n}, L_{2n}, L_{3n}$, are simple arcs having only a_0 as their common endpoint and that the set

$$B = L_{01} \cup L_{02} \cup L_{03} \cup \bigcup_{n=2}^{\infty} (L_{1n} \cup L_{2n} \cup L_{3n})$$

is a countable broom. We shall show that B is not homeomorphic to any subset of the plane E^2. First let us observe that

$$\text{(1)} \qquad \lim_{n=\infty} L_{1n} = L_{01} \cup L_{02},$$

$$\text{(2)} \qquad \lim_{n=\infty} L_{2n} = L_{01} \cup L_{03},$$

$$\text{(3)} \qquad \lim_{n=\infty} L_{3n} = L_{02} \cup L_{03},$$

where the limit of sets is taken in the sense of Hausdorff.

Now let us suppose that there exists a homeomorphism h mapping B onto a subset of the plane E^2, given in E^3 by the equation $z = 0$. It is easy to see that there exists a homeomorphism g of E^2 onto itself which is inverse to h on the set

$$T = L_{01} \cup L_{02} \cup L_{03},$$

i. e. it satisfies the condition

$$gh(p) = p \quad \text{for every point } p \in T.$$

It follows that replacing h by gh we can assume at once that

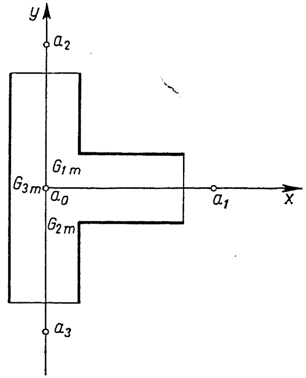
$$\text{(4)} \qquad h(p) = p \quad \text{for every point } p \in T.$$

Now let us denote by G_{1m}, to each $m = 2, 3, \ldots$, the domain in E^2 consisting of all points $(x, y, 0)$ with $0 < x < 1/m$ and $0 < y < 1 - 1/m$, or with $0 < x < 1 - 1/m$ and $0 < y < 1/m$. Manifestly the boundary of G_{1m} is a union of 6 segments; two of them, which start at the point $(1/m, 1/m, 0)$, will be said to be *main segments on the boundary of* G_{1m}.

Similarly, let us denote by G_{2m} the domain in E^2 consisting of all points $(x, y, 0)$ with $0 < x < 1/m$ and $-1+1/m < y < 0$, or with $0 < x < 1 - 1/m$ and $-1/m < y < 0$. The boundary of G_{2m} is a union of 6 segments, two of which start at the point $(1/m, -1/m, 0)$. They will be said to be *main segments on the boundary of* G_{2m}.

Finally, let us denote by G_{3m} the domain in E^2 consisting of all points $(x, y, 0)$ with $-1/m < x < 0$ and $-1+1/m < y < 1-1/m$. The segment with endpoints $(-1/m, -1+1/m, 0)$ will be called *main segment on the boundary of* G_{3m}.

Since the common part of T with each of the arcs L_{in} $(i = 1, 2, 3;$ $n = 2, 3, \ldots)$ consists only of the point a_0, we infer by (1), '(2) (3) and (4) that for every $m = 2, 3, \ldots$ there exists an index $N(m)$ such that for every $n > N(m)$ three following conditions are satisfied:



1° The simple arc $h(L_{1n})$ contains a simple arc L'_{1n}, whose interior is included in G_{1m} and one endpoint lies on the segment

$$\overline{(0, 1-1/m, 0)\ (1/m, 1-1/m, 0)},$$

while the other lies on the segment

$$\overline{(1-1/m, 0, 0)\ (1-1/m, 1/m, 0)}.$$

2° The simple arc $h(L_{2n})$ contains a simple arc L'_{2n}, whose interior is included in G_{2m} and endpoints lie on the segments

$$\overline{(0, -1+1/m, 0)\ (1/m, -1+1/m, 0)}$$

and

$$\overline{(1-1/m, 0, 0)\ (1-1/m, -1/m, 0)}$$

respectively.

3° The simple arc $h(L_{3n})$ constains as imple arc L'_{3n}, whose interior is included in G_{3m} and endpoints lie on the segments

$$\overline{(0, 1-1/m, 0)(-1/m, 1-1/m, 0)} \text{ and } \overline{(0, -1+1/m, 0)(-1/m, -1+1/m, 0)},$$

respectively.

Now let L be a simple arc in E^2 having with T only the point a_0 in common. It is easily seen that for every sufficiently large index n, there exists in L a subarc L' whose interior lies in one of the domains G_{im}, $i = 1, 2$ or 3, and joins a_0 with a point belonging to one of the main segments on the boundary of G_{im}. It follows, by 1°, 2° and 3°, that for every $n > N(m)$ the arcs L' and $L'_{in} \subset h(L_{in})$ have at least one point distinct from a_0 in common. Consequently for n sufficiently large, the simple arc $h(L_{12})$ intersects one of the arcs $h(L_{1n}), h(L_{2n}), h(L_{3n})$, in a point $\neq a_0$. But this is impossible, because h is a homeomorphism and for $n > 2$ the arcs L_{1n}, L_{2n}, L_{3n}, have with the arc L_{12} only the point $a_0 = h(a_0)$ in common.

Thus the proof of the theorem is complete.

REFERENCES

[1] J. Charatonik, *On ramification points in the classical sense*, Fundamenta Mathematicae 51 (1962), p. 229-252.

[2] A. Lelek, *On plane dendroids and their end points in the classical sense*, ibidem 49 (1961), p. 301-319.

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ON CANTORIAN MANIFOLDS IN A STRONGER SENSE

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Modifying the original definition of Cantorian manifolds, given by Urysohn in 1925, Alexandroff determined in 1957 (see [1] or § 1 below) a class of compacta that will be called here Cantorian manifolds in the stronger sense. The question has recently been raised by Borsuk whether every Cantorian manifold which is an ANR-set is a Cantorian manifold in the stronger sense. In the present note we answer this question in the affirmative for the 2-dimensional case (see § 3), and find a 3-dimensional counter-example (see § 4). Related topics are also examined.

§ 1. Four kinds of Cantorian manifolds. Roughly speaking, Cantorian manifolds are compacta whose separators have large dimensions. We recall that a set S is said to be a *separator* of the space X between the sets A and B if there exists a decomposition $X-S = M\cup N$ such that $\overline{M}\cap N = 0 = M\cap \overline{N}$, $A\subset M$ and $B\subset N$.

Let X be a *compactum*, i. e. compact metric space. Following Alexandroff (see [1], p. 70), for every integer n, we consider the condition:

($\mathbf{U}^n$) *If $A, B\subset X$ are closed sets containing interior points, then every closed separator S of X between A and B satisfies*

$$n-1 \leqslant \dim S.$$

Evidently, condition ($\mathbf{U}^n$) is equivalent to the inequality $n \leqslant \operatorname{dc} X$ (see [5], p. 105). Since one always has $\operatorname{dc} X \leqslant \dim X$, and the Cantorian manifolds are characterized by the equality $\operatorname{dc} X = \dim X$ (ibidem), the following property (**U**) of the compactum X is necessary and sufficient for X to be a *Cantorian manifold*:

(**U**) *Condition* ($\mathbf{U}^n$) *holds for* $n = \dim X$.

Since, for compacta S, the inequality $n-1 \leqslant \dim S$ is equivalent to the inequality $0 < d_{n-1}(S)$, where $d_m(S)$ denotes the m-dimensional degree of S (see [5], p. 60), Alexandroff's modification of condition ($\mathbf{U}^n$) is the following (see [1], p. 70):

($\mathbf{V}^n$) *If $A, B\subset X$ are closed sets containing interior points, then there exists a number $\sigma > 0$ such that every closed separator S of X between*