COMMENTS ON SOME WALLACE'S PROBLEMS
ON TOPOLOGICAL SEMIGROUPS

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In [20] Wallace lists nine problems on topological semigroups (P 326-
334). This note is intended to review the present status of this latest
set, and to indicate directions in which the author feels some of the more
interesting might take. I shall state each problem and follow it with
my comments. In the following, semigroup will always mean topological
semigroup (i.e., a Hausdorff space with a continuous associative
multiplication). We shall use $S$ to denote the semigroup, $E$ its set of
idempotents, and $K$ the kernel (minimal ideal) when it exists.

P 326. Is it possible to construct a semigroup on the closed $n$-cell,
$n > 2$, such that $E$ is the boundary?

Comments on P 326. The answer is still far from known, although
in the case $n = 2$, a number of results have been obtained, mostly by
participants in a seminar of R. J. Koch's during the past year. For
example, one can easily show that every element of $S$ has a square root,
and from this one obtains (using the methods of A. Lester Hudson [6])
that every element lies on an $I$-semigroup with end points on the boundary.
Further properties can be established using these subsemigroups. Again,
in [6] it is shown that the boundary of $S$ cannot be a subsemigroup, for
this implies the existence of idempotents in the interior.

P 327. Is it possible to construct a continuous associative multi-
plification on an $n$-sphere in such a way that (i) every element is the
product of two elements, (ii) there is a zero element.

Comments on P 327. It is generally conjectured that there is
no non-trivial semigroup on a sphere $X$ with $X^2 = X$ except the groups
on $S^1$ and $S^2$. In dimension 3, this was proved by Koch and Wallace [11].
If there exists a structure with non-trivial multiplication (i.e., not such
that $xy = x$ or $xy = y$ for $x, y \in X$ such that $X^2 = X$, one can show that
there exists one with zero, so that the problem is mere general than it
appears. It has been shown by Mostert and Shields [16] that if $X = S^1$
has a non-trivial connected subgroup, then $X^3 \neq X$. Wallace's genera-
lization [21] of a portion of that paper, together with the further techniques of [16] prove the following: If $X$ is an $n$-sphere with a subgroup $H_n$ which has a subset cutting $X$, then either $H_n = X$, or $H_n$ cuts $X$, and $eS$ is an $L$-semigroup [14] with zero whose boundary is $H_n$. (Hence $H_n$ is an $n-1$ sphere and thus either $n = 2$ or $n = 4$). Further $X^2 \neq X$. The full known structure can be read from [16] by simply reading "$S^m$" for "$S^m$" and "$S^n$" for "circle". It is still unknown whether there is a relatively simple description for $S = eS$ beyond that given.

**P 326.** If $G$ is a compact totally disconnected metrizable group, does there exist a tree (compact, connected, locally connected, acyclic, one-dimensional, metrizable space) which is a semigroup $S$ with identity such that the maximal subgroup is precisely $G$ which is also the set of end-points of $S$?

**Comments on P 326.** This has been solved in the affirmative by Hunter and Rothman [8]. One takes the unit interval $[0,1]$ under the usual multiplication (or any other with $0$ acting as a zero and $1$ the identity). Let $a_i ightarrow 1$, $i = 0, 1, \ldots$, be a convergent sequence with $a_0 = 0$ and $H_{a_i}, H_{a_i}, \ldots$ be normal closed subgroups of $G$ such that $H_{a_i} = G \neq G_{a_i}$ and $G/H_{a_i}$ is finite, $i = 0, 1, 2, \ldots$. One then takes $[0,1]$/$G$ modulo the following equivalence relation: $(a, g) \sim (a', g')$ if $a = a' \neq 0$, and $g' = gH_{a_i}$ whenever $a_i < a < a_i (0, g) \sim (0, g')$ for all $g, g'$.

The problem here is slightly more restrictive than Wallace’s statement and the example is hence also a solution of that statement. In the case of the Cantor group (and another closely related question solved by Koch and McAuley – unpublished) the answer was already known.

**P 329.** Suppose $S$ is a semigroup with identity which is topologically Euclidean $n$-space. Can compact connected subgroups containing the identity be self-linked?

**Comments on P 329.** For the special case $n = 3$, and any compact subgroup $G$ (not necessarily containing the identity), Curtis [1] proved the statement in the negative. The author, with a simpler proof, has shown that there can be no self-linked subgroups of any contractible semigroups, and the existence of an identity is not assumed [12].

**P 330.** If $S$ is a compact connected locally connected metrizable one-dimensional semigroup with identity, then it is known that $S$ is either a dendrite or contains exactly one simple closed curve which coincides with the minimal ideal of $S$. Is there an analogous proposition for higher dimensions?

**Comments on P 330.** That anything analogous occurs in higher dimensions would seem remote—both in probability and closeness of the analogy. Certainly it would seem impossible to prove that there are a small number of cases where $S$ is, say, acyclic, and a small number of cases where it is not, except in dimension one. Some of the interesting examples of Hunter [9] and Hunter and Rothman [10] show the diversity even in dimension two.

**P 331.** If $S$ is a compact connected commutative semigroup with identity all of whose elements are idempotent, does $S$ have the fixed point property?

**Comments on P 331.** In a conversation with Wallace, J. L. Kelley pointed out an unpublished result of his which solves this in the affirmative in case $S$ is finite dimensional and locally connected. Kelley’s theorem is as follows:

Let $X$ be a finite dimensional locally connected continuum. If there is a retract $f$ of $X \times X$ onto the diagonal such that $f(0, y) = f(0, y, 0)$, then $X$ is an absolute retract.

The general situation is still open despite the author’s statement to the contrary in his review of Wallace’s problems for the American Mathematical Reviews.

**P 332.** Let $S$ be a compact connected semigroup, and $E$ the set of idempotents. If $S = SSE$, does the minimal ideal $K$ and $S$ have the same cohomology?

**Comments on P 332.** If $S$ has an identity, this is known to be true [13], and a number of even weaker conditions are sufficient. However, recently A. Lester Hudson has constructed an example of a semigroup with zero and $SSE = E$ on a 2-sphere with four “whiskers” at the zero [3].

**P 333.** It is a corollary to the result of P 332 that a compact connected semigroup with zero and identity is uniformalnt. Is there a proof of this using only set-theoretic topology?

**Comments on P 333.** As of the moment, no such proof is known.

**P 334.** Let $S$ be a compact semigroup and let $B$ denote the “boundary” of $S$ in some suitable sense.

(a) If every element of $S$ has a square-root in $S$ does every element of $B$ have a square root in $B$ (Problem of H. H. Corson).

(b) Under some interpretations of “boundary” it is known that if $S$ has an identity, then it lies in $B$. Are there other useful interpretations of “boundary” for which this is so?

(c) If one assumes multiplication on $B$ is commutative, are there agreeable conditions under which it may be shown to be commutative on $S$?

**Comments on P 334.** (a) has still not been investigated formally and shows promise of interesting and important consequences. There
are a number of incidental results that give examples in support of the
conjecture that this is so — at least if the boundary is regular [14], and
[9].

It would seem that (b) has already been solved in its best possible
form by Mostert and Shields in [15], for there it is solved for relative
manifolds with boundary (i.e., $S$ compact, $B$ a closed set such that $S\setminus B$
is locally Euclidean), and since the proof only used cohomology proper-
ties of such spaces, it is true for spaces $(S, B)$ where $S$ is compact, $B$
closed, and $S\setminus B$ a cohomology manifold. It would be difficult to conceive
of a more general form of “boundary”.

In special cases, most notably in [15], (c) is known to be true. How-
ever, the general problem seems not yet to have been studied.

One of the most useful tools so far in virtually all the questions
concerning the boundary (and also in boundary-like objects [2], [3]) has
been the construction of one-parameter subsemigroups [18]. However,
in the questions posed above, there seem definite limitations to their
use, and it would appear other strong tools will be needed and devel-
oped in their solutions. Actually, it appears that it is not “boundary”
that is important in these questions, but the existence of certain dis-
tinguished subsets which are sufficiently large (boundary-like for certain
open subsets) [2, 3, 4, 5, 6, 7, 12, 14, 15, 16, 17, 21] and this is usu-
ally the boundary in case $S$ is compact [6, 14]. The compactness restric-
tion is not always necessary even for this distinguished set [4, 5, 12],
but certainly makes life easier. In this connection, one has the problem of
Mostert and Shields [15], if $S$ is a semigroup with identity on a manifold,
and $L$ is the boundary of the maximal connected subgroup $G$ (which is open [15]), does $L$ contain an idempotent? This is solved if $S$ is the plane, but is unknown other-

REFERENCES

[2] K. H. Hofmann, Locally compact semigroups in which a subgroup with
compact complement is dense, Transactions of the American Mathematical
[3] — Homogeneous locally compact groups with compact boundary, ibidem