Let s $(1 \leqslant s \leqslant q)$ be chosen so that $j_s \neq i_1, \ldots, i_p$. Then b_{j_s} differs from the elements $b_{i_1}, \ldots, b_{i_p}$ and

$$b_{i_1} \cap \ldots \cap b_{i_p} \leqslant b_{j_s}.$$

Hence, taking $p = r-1$, $a_k = b_{i_k}$ $(1 \leqslant k \leqslant r-1)$ and $a_r = b_{j_s}$, we find (3) satisfied.

Conversely, suppose that there exist elements $a_1, \ldots, a_r$ in $T(\subseteq S)$ such that (3) holds. Then

$$f^{(r)}_{1,\ldots,r}(a_1, \ldots, a_r) = f^{(r)}_{1,\ldots,r-1}(a_1, \ldots, a_r).$$

Since this equation does not hold identically, T is M-dependent.

Thus Theorem 6 and its Corollary are proved.

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CONCERNING THE INDEPENDENCE IN LATTICES

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The independence is meant here in the sense of [2] and [3]. The results presented here complete the paper [4] by Szász, in particular Theorem 1 is a strengthening of Theorem 3 of [4].

Nevertheless, the knowledge of Szász' paper is not necessary for the reader of this note.

The proof of Theorem 1 is a modification of Szasz' proof, made by J. Płonka.

1. Let us consider a lattice $(L; \cup, \cap)$.

THEOREM 1. *If I is a set of independent elements of L, then*

(i) $a_1 \cap \ldots \cap a_m$ non $\leqslant b_1 \cup \ldots \cup b_n$ *for each sequence* $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ $(m \geqslant 1, n \geqslant 1)$ *of different elements of* L (1).

Proof. Let us suppose

$$\bigcap_{j=1}^{m} a_j \leqslant \bigcup_{j=1}^{n} b_j$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n$ is a sequence of different elements of L. Hence

$$(*) \qquad \bigcap_{j=1}^{m} a_j \cup \bigcup_{j=1}^{n} b_j = \bigcup_{j=1}^{n} b_j.$$

Let us consider the following algebraic operations in L (= lattice polynomials):

$$f(x_1, \ldots, x_m, y_1, \ldots, y_n) = \bigcap_{j=1}^{m} x_j \cup \bigcup_{j=1}^{n} y_j,$$

and

$$g(x_1, \ldots, x_m, y_1, \ldots, y_n) = \bigcup_{j=1}^{n} y_j.$$

(1) The condition (i) for sets has been formulated by Tarski [5], p. 61. In this case (i) is equivalent to a condition treated in [3], p. 141, theorem (iii).

Let us put $a = a_1 \cup b_1$ and $b = a_1 \cap b_1$. Since $a_1 \neq b_1$, we have $a > b$. Hence

$$f(a, \ldots, a, b, \ldots, b) = a \neq b = g(a, \ldots, a, b, \ldots, b),$$

and consequently $f \neq g$.

On the other hand, the equality (*) can be written as follows:

$$f(a_1, \ldots, a_m, b_1, \ldots, b_n) = g(a_1, \ldots, a_m, b_1, \ldots, b_n),$$

and hence $a_1, \ldots, a_m, b_1, \ldots, b_n$ are dependent.

Theorem 1 is thus proved.

2. For any family F of subsets of the set of indices $N = \{1, 2, \ldots, n\}$ we define an operation in L:

$$p_F(x_1, \ldots, x_n) = \bigcup_{S \in F} \bigcap_{j \in S} x_j.$$

It is easy to see that for every family F there exists a subfamily F_0 of sets incomparable by the inclusion (in the sequel we shall briefly say: incomparable) and such that $p_F = p_{F_0}$.

For every F the operation p_F is algebraic in L. It is known that the converse implication is true under the hypothesis that L is distributive ([1], p. 145, Theorem 12).

The following lemma concerning the incomparable sets will be used in the next section:

LEMMA. *If F and G are two different families of incomparable sets, then either* 1° *there exists such a set $S_0 \in F$ that $T \setminus S_0 \neq 0$ for every $T \in G$, or* 2° *there exists such a set $T_0 \in G$ that $S \setminus T_0 \neq 0$ for every $S \in F$.*

Proof. Let us suppose that none of the conditions 1° and 2° is satisfied. Consequently, if $S \in F$, then there exists a set $T \in G$ and a set $S_1 \in F$ such that $S \supset T \supset S_1$. Since F is a family of incomparable sets, we have $S = S_1$, whence $S = T$ and consequently $S \in G$. Thus $F \subset G$ and, by symmetry, $G \subset F$. Therefore $F = G$, which contradicts the hypothesis.

3. Now we can prove the converse of Theorem 1 for distributive lattices:

THEOREM 2. *If a lattice L is distributive, then each subset I of L satisfying the condition* (i) *is a set of independent elements.*

Proof. Let us suppose that I is a set of dependent elements. Thus, there exists a sequence $a_1, \ldots, a_n$ of different elements of I and such two different families F and G of incomparable subsets of the set $N = \{1, 2, \ldots, n\}$ that

$$p_F(a_1, \ldots, a_n) = p_G(a_1, \ldots, a_n).$$

Applying the Lemma, we may admit by symmetry, that there exists a set $S_0 \in F$ such that $T \setminus S_0 \neq 0$ for every $T \in G$. Consequently

$$\bigcap_{j \in S_0} a_j \leqslant p_F(a_1, \ldots, a_n) = p_G(a_1, \ldots, a_n) \leqslant \bigcup_{j \in N \setminus S} a_j,$$

whence I does not satisfy (i).

Theorem 1 and 2 imply the

COROLLARY 3. *If L_0 is a sub-lattice of a distributive lattice L then a subset I of L_0 is a set of independent elements in L_0 if and only if it is so in L.*

4. We shall prove finally that *the distributivity is essential in Theorem* 2 *and in Corollary* 3. In fact, let L_0 be a distributive lattice containing at least three independent elements a, b, c, and L a non-distributive lattice containing L_0. Since $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$, the elements a, b, c, are dependent in L. Consequently, Corollary 3, and therefore Theorem 2, are not valid in L.

As the lattice L_0 we may admit e. g. the class of all subsets of the set $\{1, 2, 3, 4, 5, 6\}$, with union and intersection as join and meet respectively. The sets $a = \{1, 2, 4\}$, $b = \{1, 3, 5\}$, and $c = \{2, 3, 6\}$, are independent in L_0.

As the lattice L we may admit the Cartesian product of L_0 and any non-distributive lattice.

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