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TRANSLATIONS OF INFINITE SUBSETS OF A GROUP

BY

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The following problem was proposed by Jan Mycielski [2]:

Let R be the real axis, $A = R - B$ and both A and B infinite. Does there exist a translation γ such that $A \cap \gamma B$ is infinite?

This problem was answered affirmatively by P. Lax ([3], p. 646). His solution is contained in Theorem 1, below (1). This paper answers the same problem for a wide class of groups, R, including all Abelian groups.

Notation. If G is a group and A is a subset of G, $G(A)$, $\tilde{A}$ and $|A|$ will denote, respectively, the subgroup generated by A, the complement of A, and the cardinality of A. $\emptyset$ will denote the empty set. $Z(G)$ is the center of G and $[G:K]$ is the index of the subgroup K in G.

Definition. A group, G, is *completely regular* (resp. *regular*) if, and only if, for each infinite subset, H, of G whose complement, $\tilde{H}$, is also infinite there exists $x \in G$ such that $Hx \cap \tilde{H}$ (resp. $Hx \cap \tilde{H}$ or $xH \cap \tilde{H}$) is infinite.

Since $xH = (H^{-1}x^{-1})^{-1}$, complete regularity defined in terms of right translations as above is equivalent to complete regularity defined in terms of left translations. Also, since $x^{-1}\tilde{H} \cap H$ resp. $\tilde{H}x^{-1} \cap H$ is infinite if and only if $xH \cap \tilde{H}$ resp. $Hx \cap \tilde{H}$ is infinite, the above definitions are symmetric in H and $\tilde{H}$. Finally, it is obvious that an Abelian group is regular if and only if it is completely regular.

LEMMA. *If G is a group which possesses a subgroup K satisfying*

1) *K is infinite,*

2) *$K = G(A)$ for some subset A such that $|A| < |G|$,*

3) *$[G:K] = |G|$,*

then G is regular.

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(1) The analogous problem, where infinity is replaced by the cardinality of continuum, is answered in the negative; see Banach [1] and Sierpiński [5], [6]. [*Note of the Editors*].

Proof. Let G, K and A be as in the hypotheses and H and $\tilde{H}$ be infinite subsets of G. We may assume that $A \supseteq A^{-1}$.

Case 1. Either $|H| < |G|$ or $|\tilde{H}| < |G|$. By symmetry it suffices to consider $|H| < |G|$. Since H is infinite, $|H \cdot H^{-1}| = |H| < |G|$. Thus for some $y \in G$, $y \notin H \cdot H^{-1}$. Now, if $h \in H$ and $yh \in H$, $y = yh \cdot h^{-1} \in H \cdot H^{-1}$ which is false. Thus $yH \cap H = \emptyset$ and $yH \cap \tilde{H} = yH$, which is infinite.

Case 2. $|H| = |\tilde{H}| = |G|$. Let $\{x_\beta | \beta \in B\}$ be a set of representatives for the cosets $\{xK | x \in G\}$. Let $B_1 = \{\beta | \beta \in B$ and $x_\beta K \cap H \neq \emptyset$ and $x_\beta K \cap \tilde{H} \neq \emptyset\}$.

A. $|B_1| = |G|$. Now for each $\beta \in B_1$ there are $y_\beta, y'_\beta \in x_\beta K$ such that $y_\beta \in H$ and $y'_\beta \in \tilde{H}$. Moreover, y_β and y'_β can be selected in such a way that $y'_\beta = y_\beta a_\beta$ for some $a_\beta \in A$; for if not, then for each $a \in A$ and each $y \in x_\beta K$, $ya \in x_\beta K \cap H$ and since $G(A) = K$, $x_\beta K \subset x_\beta K \cap H$ contrary to definition of B_1. But $|B_1| > |A|$ so that for some $a \in A$, $a = a_\beta$ for infinitely many $\beta \in B_1$. $z_\beta a \in \tilde{H}$ for infinitely many $z_\beta \in H$ and $Ha \cap \tilde{H}$ is infinite.

B. $|B_1| < |G|$. Let $B_2 = \{\beta | x_\beta K \cap H = \emptyset\}$ and $B_3 = \{\beta | x_\beta K \cap \tilde{H} = \emptyset\}$. Since $B = B_1 \cup B_2 \cup B_3$ and $|B| = |G|$, at least one of B_2, B_3 is not empty. By symmetry, we may assume $B_3 \neq \emptyset$. Let $\beta_3 \in B_3$; $x_{\beta_3} K \subset H$. Now, if $\beta_2 \in B_2$, $H \supset x_{\beta_2} K = x_{\beta_2} x_{\beta_3}^{-1} x_{\beta_3} K \subset x_{\beta_2} x_{\beta_3}^{-1} H$ so that $x_{\beta_2} x_{\beta_3}^{-1} H \cap \tilde{H}$ is infinite. If, however, $B_2 = \emptyset$, then $\tilde{H} \subset \{x | x \in x_\beta K, \beta \in B_1\}$ and since $|B_1| < |G|$, and $|\tilde{H}| = |G|$, $\tilde{H} \cap x_\beta K$ is infinite for some $\beta \in B_1$. Then $x_\beta x_{\beta_3}^{-1} H \supset x_\beta K$ and $x_\beta x_{\beta_3}^{-1} H \cap \tilde{H}$ is infinite.

THEOREM 1. *Every uncountable group is regular.*

Proof. Let G be uncountable and let A be a countable subset of G and let $K = G(A)$. Since K is countable and $[G:K] = |G|$ the lemma applies.

THEOREM 2. *If G is countable and contains a finitely-generated infinite subgroup of infinite index, then G is regular.*

THEOREM 3. *If G is countable and contains an element, x, of infinite order and if $[G\colon G(x)]$ is finite, then*

1) *if $Z(G) \cap G(x) = \{e\}$, G is regular but not completely regular,*

and

2) *if $Z(G) \cap G(x) \neq \{e\}$, G is not regular.*

Proof. Since $[G\colon G(x)]$ is finite, there is an element $y \in G(x)$, $y \neq e$ such that $G(y)$ is normal in G. Furthermore, $[G\colon G(y)]$ is finite, and $Z(G) \cap G(y) = \{e\}$ if and only if $Z(G) \cap G(x) = \{e\}$ (see [4], p. 82-84). Hence, there is no loss in generality in assuming $G(x)$ is normal. Let $r_1, r_2, \dots, r_n$ be a set of representatives for the cosets of $G(x)$ in G. Now, $r_i x r_i^{-1} = x$ or $r_i x r_i^{-1} = x^{-1}$ for $i = 1, 2, \dots, n$. Let $H = \{x^m r_i | i = 1, 2, \dots, n;\ m > 0\}$. Then, if $z = x^k r_j$, $Hz = \{x^m r_i x^k r_j |\ i = 1, 2, \dots, n;\ m > 0\} = \{x^m x^{\pm k} r_i r_j | i = 1, 2, \dots, n;\ m > 0\} = \{x^m x^{\pm k} x^t r_p |\ p = 1, 2, \dots, n;\ m > 0\}$. Since k is fixed and t depends only on i and j and is bounded, $Hz \cap \tilde{H}$ is finite. This establishes the negative assertion in 1. Since, in case 2 we may assume $r_i x r_i^{-1} = x$ for all i, $zH = \{x^{k+m+t} r_p | m > 0\}$, and $zH \cap \tilde{H}$ is also finite.

It remains to prove that G is regular in 1. Let H and $\tilde{H}$ be infinite. If either $H \cap G(x)$ or $\tilde{H} \cap G(x)$ is finite (say $H \cap G(x)$), then for some r_i, $r_i G(x) \cap H$ is infinite and $r_i^{-1} H \cap \tilde{H}$ is infinite. Hence we may assume that $H \cap G(x)$ and $\tilde{H} \cap G(x)$ are both infinite and that G is not regular. If both H and $\tilde{H}$ contain arbitrarily large or arbitrarily small powers of x, $xH \cap \tilde{H}$ is infinite so we may also assume that one of H, $\tilde{H}$ contains all sufficiently large powers of x and the other all sufficiently small powers of x. We assume $H \supset \{x^n | n > m_0\}$. Since $Z(G) \cap G(x) = \{e\}$, for some r_i, $r_i x r_i^{-1} = x^{-1}$. Now, since $H r_i^{-1} \cap \tilde{H}$ is finite, $x^n r_i^{-1} \in H$ for all $n > m_1$. But then $r_i H \cap \tilde{H}$ can be finite only if $r_i x^n r_i^{-1} \in H$ for all $n > m_2$. Thus $x^{-n} \in H$ for all $n > m_2$, which is impossible since $\tilde{H}$ contains all sufficiently small powers of x. This completes the proof of theorem 3.

THEOREM 4. *If G is a countable group which is the union of an increasing sequence of finite groups, then G is not regular.*

Proof. Let $G = \bigcup_{n=0}^{\infty} G_n$ where G_n is a finite group and $G_i \subsetneq G_{i+1}$. Let $p(x) = \min\{n | x \in G_n\}$ for each $x \in G$. Let $H = \{x | p(x)$ is even$\}$. H and $\tilde{H}$ are clearly infinite. If $p(x) \neq p(y)$, $p(xy) = \max\{p(x), p(y)\}$. Thus if $y \in G$, $p(xy) = p(yx) = p(x)$ for all x such that $p(x) > p(y)$, hence for all but finitely many $x \in G$. Therefore for each $y \in G$, $Hy \cap \tilde{H}$ and $yH \cap \tilde{H}$ are finite and G is not regular.

THEOREM 5. *If G is a countable abelian group, G is regular if, and only if*

1) *G contains an element, x, of infinite order,*

and

2) *$G/G(x)$ is infinite.*

Proof. Sufficiency is given by theorem 2.

Necessity. If G does not contain an element of infinite order it satisfies the hypotheses of theorem 4, and is therefore not regular. If G contains an element, x, of infinite order such that $G/G(x)$ is finite it satisfies the hypotheses of theorem 3, 2) and is not regular.

P 418. Is there an uncountable group which is not completely regular?

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COMMENTS ON SOME WALLACE'S PROBLEMS ON TOPOLOGICAL SEMIGROUPS

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In [20] Wallace lists nine problems on topological semigroups (P 326--334). This note is intended to review the present status of this latest set, and to indicate directions in which the author feels some of the more interesting might take. I shall state each problem and follow it with my comments. In the following, *semigroup* will always mean *topological semigroup* (i. e., a Hausdorff space with a continuous associative multiplication). We shall use S to denote the semigroup, E its set of idempotents, and K the kernel (minimal ideal) when it exists.

P 326. Is it possible to construct a semigroup on the closed n-cell, $n \geqslant 2$, such that E is the boundary?

Comments on P 326. The answer is still far from known, although in the case $n = 2$, a number of results have been obtained, mostly by participants in a seminar of R. J. Koch's during the past year. For example, one can easily show that every element of S has a square root, and from this one obtains (using the methods of A. Lester Hudson [6]) that every element lies on an I-semigroup with end points on the boundary. Further properties can be established using these subsemigroups. Again, in [6] it is shown that the boundary of S cannot be a subsemigroup, for this implies the existence of idempotents in the interior.

P 327. Is it possible to construct a continuous associative multiplication on an n-sphere in such a way that (i) every element is the product of two elements, (ii) there is a zero element.

Comments on P 327. It is generally conjectured that there is no non-trivial semigroup on a sphere X with $X^2 = X$ except the groups on S^1 and S^3. In dimension 1, this was proved by Koch and Wallace [11]. If there exists a structure with non-trivial multiplication (i. e., not such that $xy = x$ or $xy = y$ for $x, y \in X$ such that $X^2 = X$, one can show that there exists one with zero, so that the problem is more general than it appears. It has been shown by Mostert and Shields [16] that if $X = S^2$ has a non-trivial connected subgroup, then $X^2 \neq X$. Wallace's genera-