TRANSLATIONS OF INFINITE SUBSETS OF A GROUP

BY

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The following problem was proposed by Jan Mycielski [2]:

Let \( R \) be the real axis, \( A = R - B \) and both \( A \) and \( B \) infinite. Does there exist a translation \( \gamma \) such that \( A \sim \gamma B \) is infinite?

This problem was answered affirmatively by P. Lax ([3], p. 646). His solution is contained in Theorem 1, below (\(^1\)). This paper answers the same problem for a wide class of groups, \( G \), including all Abelian groups.

Notation. If \( G \) is a group and \( A \) is a subset of \( G \), \( G(A) \), \( \tilde{A} \) and \( |A| \) will denote, respectively, the subgroup generated by \( A \), the complement of \( A \), and the cardinality of \( A \). \( \varepsilon \) will denote the empty set. \( Z(G) \) is the center of \( G \) and \( [G : K] \) is the index of the subgroup \( K \) in \( G \).

Definition. A group, \( G \), is completely regular (resp. regular) if, and only if, for each infinite subset, \( H \), of \( G \) whose complement, \( \tilde{H} \), is also infinite there exists \( x \in G \) such that \( H \cup \tilde{H} \) (resp. \( H \cap \tilde{H} \) or \( x H \cap \tilde{H} \)) is infinite.

Since \( a H = (H - a^{-1})^{-1} \), complete regularity defined in terms of right translations as above is equivalent to complete regularity defined in terms of left translations. Also, since \( x^{-1} H \cap \tilde{H} \) resp. \( H \cap \tilde{H} \) is infinite if and only if \( xH \cap \tilde{H} \) resp. \( H \cap \tilde{H} \) is infinite, the above definitions are symmetric in \( H \) and \( \tilde{H} \). Finally, it is obvious that an Abelian group is regular if and only if it is completely regular.

Lemma. If \( G \) is a group which possesses a subgroup \( K \) satisfying

1) \( K \) is infinite,

2) \( K = G(A) \) for some subset \( A \) such that \( |A| < |G| \),

3) \( [G : K] = |G| \),

then \( G \) is regular.

\(^1\) The analogous problem, where infinity is replaced by the cardinality of continuum, is answered in the negative; see Banach [1] and Sierpiński [5], [6]. [Note of the Editor].
Proof. Let $G$, $K$ and $A$ be as in the hypotheses and $H$ and $\bar{H}$ be infinite subsets of $G$. We may assume that $A = A^2$.

Case 1. Either $|H| < |G|$ or $|\bar{H}| < |G|$. By symmetry it suffices to consider $|H| < |G|$. Since $H$ is finite, $|H^2| = |H| < |G|$; thus for some $y \in G$, $y \neq H^{-1}$ and $y^k \neq H^{-1}$. Now, if $k \neq H$ and $y \neq H^{-1}$ then $y = y^k \neq H^{-1}$, which is false. Thus $yH \cap H = \emptyset$ and $yH \cap \bar{H} = \emptyset$, which is infinite.

Case 2. $|H| = |\bar{H}| = |G|$. Let $(a \beta | \bar{e} B)$ be a set of representatives for the cosets $(aK | \bar{e} K \cdot G)$. Let $B_1 = \{\beta | \bar{e} B_1 = a \bar{K} \cap H \neq \emptyset \}$ and $B_2 = \{\beta | \bar{e} B_2 = \bar{K} \cap \bar{H} \neq \emptyset \}$.

A. $|B_1| = |G|$. Now for each $\beta \in B_1$ there are $y_\beta, y_\beta', \bar{e} \beta K$ such that $y_\beta H = y_\beta' H$ and $y_\beta' H$ and $y_\beta'$ can be selected in such a way that $y_\beta = y_\beta \bar{e} \delta_1$ for some $\delta_1 \in A$; and in fact, they are unique. Thus $\alpha \bar{K} H \cap \bar{H}$ is infinite. Hence, we may assume that $H = \{\bar{e} \delta_1 \bar{e} K \cap H \}$ and that $G$ is not regular. If both $H$ and $\bar{H}$ contain arbitrarily large or arbitrarily small powers of $x$, $xH \cap \bar{H}$ is infinite, so we may also assume that one of $H$, $\bar{H}$ contains all sufficiently large powers of $z$ and the other all sufficiently small powers of $z$. We assume $H = \{\bar{e} \delta_1 m > m_0 \}$ and $\bar{H} = \{\bar{e} \delta_1 m < m_1 \}$. Since $\bar{e} \delta_1 \bar{e} K \cap H$ is finite, for some $\bar{e} \delta_1 \bar{e} K \cap \bar{H}$ is finite and $\{\bar{e} \delta_1 \bar{e} K \cap \bar{H} \}$ is infinite.

Now, $H^2 = \{\bar{e} \delta_1 \bar{e} K \cap \bar{H} \}$ and $\bar{H}^2 = \{\bar{e} \delta_1 \bar{e} K \cap H \}$, and since $|B_1| < |G|$, $H \cap \bar{H}$ is infinite for some $\beta \in B_1$. Then $\bar{e} \delta \bar{e} \delta_1 \bar{e} K \cap \bar{H}^2$ is infinite.

Theorem 1. Every uncountable group is regular.

Proof. Let $G$ be uncountable and let $A$ be a countable subset of $G$ and let $K = G(A)$. Since $K$ is countable and $[G : K] = |G|$ the lemma applies.

Theorem 2. If $G$ is countable and contains a finitely-generated infinite subgroup of infinite index, then $G$ is regular.

Theorem 3. If $G$ is countable and contains an element $x$, of infinite order and if $G : [G : \bar{e} x]$ is finite, then

1) $\bar{e} x \bar{e} xG \cap G = \{e\}$, $G$ is regular but not completely regular, and

2) if $\bar{e} x \bar{e} xG \cap G = \{e\}$, $G$ is not regular.

Proof. Since $G : [G : \bar{e} x]$ is finite, there is an element $y \in G \cdot \bar{e} x$ such that $y \neq G$ is normal in $G$. Furthermore, $[G : \bar{e} x(\bar{e} x)]$ is finite, and $Z(G) \cdot \bar{e} x \bar{e} x = \{e\}$ if and only if $Z(G) \cdot \bar{e} x(\bar{e} x) = \{e\}$ (see [4] p. 83-84). Hence, there is no loss in generality in assuming $G(x)$ is normal. Let $r_1, r_2, \ldots, r_n$ be a set of representatives for the cosets of $G(x)$ in $G$.

Now, $r_i \bar{e} x \bar{e} x = x$ or $r_i \bar{e} x \bar{e} x = x^{-1}$, for $i = 1, 2, \ldots, n$. Let $H = \{r_i \bar{e} x \bar{e} x \mid i = 1, 2, \ldots, n \}$ and $\bar{H} = \{r_i \bar{e} x \bar{e} x \mid i = 1, 2, \ldots, n \}$. Then, $H \cap \bar{H} = \emptyset$, $H = \{r_i \bar{e} x \bar{e} x \mid i = 1, 2, \ldots, n \}$ and $\bar{H} = \{r_i \bar{e} x \bar{e} x \mid i = 1, 2, \ldots, n \}$.

REFERENCES


COMMENTS ON SOME WALLACE'S PROBLEMS ON TOPOLOGICAL SEMIGROUPS

by

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In [20] Wallace lists nine problems on topological semigroups (P 326-334). This note is intended to review the present status of this latest set, and to indicate directions in which the author feels some of the more interesting might take. I shall state each problem and follow it with my comments. In the following, semigroup will always mean topological semigroup (i.e., a Hausdorff space with a continuous associative multiplication). We shall use $S$ to denote the semigroup, $B$ its set of idempotents, and $E$ its kernel (minimal ideal) when it exists.

P 326. Is it possible to construct a semigroup on the closed $n$-cell, $n \geq 2$, such that $E$ is the boundary?

Comments on P 326. The answer is still far from known, although in the case $n = 2$, a number of results have been obtained, mostly by participants in a seminar of R. J. Koch's during the past year. For example, one can easily show that every element of $S$ has a square root, and from this one obtains (using the methods of A. Lester Hudson [8]) that every element lies on an $I$-semigroup with end points on the boundary. Further properties can be established using these subsemigroups. Again, in [6] it is shown that the boundary of $S$ cannot be a subsemigroup, for this implies the existence of idempotents in the interior.

P 327. Is it possible to construct a continuous associative multiplication on an $n$-sphere in such a way that (i) every element is the product of two elements, (ii) there is a zero element.

Comments on P 327. It is generally conjectured that there is no non-trivial semigroup on a sphere $X$ with $X^2 = X$ except the groups on $S^1$ and $S^3$. In dimension 1, this was proved by Koch and Wallace [11]. If there exists a structure with non-trivial multiplication (i.e., not such that $xy = x$ or $xy = y$ for $x, y \in X$ such that $X^2 = X$, one can show that there exists one with zero, so that the problem is mere general than it appears. It has been shown by Mostert and Shields [16] that if $X = S^1$ has a non-trivial connected subgroup, then $X^2 \neq X$. Wallace's genera-