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RELATIONS BETWEEN NETS AND INDEXED FILTER BASES*

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In a preceding note [1], the present writer indicated a relation between the convergence theories of nets and filter bases and showed how these theories can be used simultaneously (see also [2, 3, 4, 5]). It was seen that nets generate unique filter bases in a natural way and that filter bases generate (non-unique) nets. A crucial construction was to show that a refinement of the filter base generated by a net gives rise to a subnet of the given net. Recently, Professor Albert Wilansky pointed out to the author that our construction fails if the original net is sufficiently degenerate. It is not difficult to present a new construction which corrects this error within the framework of the original note. However, we prefer to supply here a different construction which has the advantage of associating a *unique* net from a filter base. We also introduce a notion of equivalence for nets, which is finer than that employed by Smiley [6], in such a way that the operations of forming "indexed" filter bases from nets and nets from indexed filter bases are inverse operations (to within equivalence).

Preliminaries. We recall that a *net* in a set X is a mapping α of a non-empty directed set A into X and that a net $\beta: B \to X$ is a *subnet* of α: $A \to X$ in case there exists a mapping π: $B \to A$ with $\beta = \alpha \circ \pi$ and satisfying the requirement that for each a_0 in A there is a b_0 in B such that if $b \geqslant b_0$, then $\pi(b) \geqslant a_0$. If β is a subnet of α, we shall write $\alpha \leqslant \beta$. Further, we shall say that two nets α, β are *equivalent* and write $\alpha \sim \beta$ in case each is a subnet of the other (it being easily verified that this is an equivalence relation). It will be noted that this relation of equivalence between nets differs from that introduced by Smiley [6]. We shall see later that if two nets are equivalent in our sense, then they are equivalent in his sense, but not conversely.

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A *filter base* $\mathscr{B}$ in a set X is a non-empty collection of non-empty subsets of X with the property that the intersection of every pair of sets in $\mathscr{B}$ contains a set in $\mathscr{B}$. A *filter* $\mathscr{F}$ in X is a filter base in X which contains all supersets. If one adjoins all supersets to a filter base $\mathscr{B}$, one obtains a unique filter, called the *filter generated by* $\mathscr{B}$. If $\mathscr{A}, \mathscr{B}$ are filter bases in X, we say that $\mathscr{B}$ is a *refinement* of $\mathscr{A}$ and write $\mathscr{A} \leqslant \mathscr{B}$ in case every element of $\mathscr{A}$ contains some element of $\mathscr{B}$. Two filter bases $\mathscr{A}, \mathscr{B}$ are *equivalent* if each is a refinement of the other in which case we write $\mathscr{A} \sim \mathscr{B}$. If $\mathscr{F}, \mathscr{G}$ are the filters generated by the filter bases $\mathscr{A}, \mathscr{B}$, then $\mathscr{A} \leqslant \mathscr{B}$ if and only if $\mathscr{F} \leqslant \mathscr{G}$ which holds if and only if the inclusion $\mathscr{F} \subseteq \mathscr{G}$ holds. Hence, two filters are equivalent if and only if they are equal.

It is not conventional to require the elements of a filter base to be indexed, but we shall find it convenient to do so. In fact, by an *indexed filter base* in X we understand a mapping $\mathscr{B}$ of a non-empty directed set B into the set of non-empty subsets of X such that if $b \leqslant b'$, then $\mathscr{B}(b) \supseteq \mathscr{B}(b')$. It is plain that every filter base $\mathscr{B}$ forms an indexed filter base in a natural way: the elements of the directed set B are the sets in $\mathscr{B}$ ordered by inclusion $\supseteq$ and the map $\mathscr{B}$ is the identity mapping. It is also clear that there are many other ways a filter base can be indexed. Clearly, an indexed filter base is a special kind of net in the power set $\mathscr{P}(X)$; alternatively, a filter base is merely the range of an indexed filter base. We shall use the same definition for refinement and equivalence of indexed filter bases as for ordinary filter bases. It is emphasized that if $\mathscr{A} \leqslant \mathscr{B}$, then it need not be the case that, considered as nets in $\mathscr{P}(X)$, $\mathscr{B}$ is a subnet of $\mathscr{A}$; indeed, the range of $\mathscr{B}$ need not be contained in the range of $\mathscr{A}$.

Henceforth it will be assumed that all filter bases are indexed, since this can always be attained. Nevertheless, we shall not drop the modifier "indexed", so that the reader will be reminded of this requirement.

The generation of nets and indexed filter bases. Let $\alpha\colon A \to X$ be a net, let $a \in A$, and let $E_a = \{\alpha(t) : t \in A,\ a \leqslant t\}$. Then it is readily seen that the map $a \to E_a$ of A into $\mathscr{P}(X)$ is an indexed filter base. This indexed filter base is called the *indexed filter base generated by the net* α and will be denoted by $\boldsymbol{B}(\alpha)$. We note that the domains of α and $\boldsymbol{B}(\alpha)$ are the same directed set A.

1. PROPOSITION. *Let α, β be nets in X.*

(i) *If $\alpha \leqslant \beta$, then $\boldsymbol{B}(\alpha) \leqslant \boldsymbol{B}(\beta)$.*

(ii) *If $\alpha \sim \beta$, then $\boldsymbol{B}(\alpha) \sim \boldsymbol{B}(\beta)$.*

The proof of this remark is very simple and will be omitted. It follows from (ii) that our definition of equivalence of nets implies Smiley's, since in effect he defines two nets to be equivalent if and only if the generated filter bases are equivalent.

The reader should see that the converse of (i) does not hold, in general, since there is no reason that the range of β be contained in the range of α. Even when this is the case there may be no suitable mapping of B into A. For example, let A be the directed set of natural numbers and $\alpha(n) = (-1)^n$ and let $B = \{1\}$ and $\beta(1) = 1$. Then $\boldsymbol{B}(\alpha) \leqslant \boldsymbol{B}(\beta)$, while β is not a subnet of α, even though the range of β is contained in the range of α.

If $\mathscr{A}\colon A \to \mathscr{P}(X)$ is an indexed filter base, then the set $\{(x, a)\colon x \in \mathscr{A}(a),\ a \in A\}$ is a directed set under the ordering $(x, a) \leqslant (x', a')$ if and only if $a \leqslant a'$. The mapping which sends (x, a) into x is a net in X and is said to be the *net* $\boldsymbol{N}(\mathscr{A})$ *generated by* $\mathscr{A}$. Note that although the domain of $\mathscr{A}$ and $\boldsymbol{N}(\mathscr{A})$ are not the same, they are closely related.

The inversion relation. We shall now show that, up to equivalence, the operations $\boldsymbol{B}$ and $\boldsymbol{N}$ are inverses of each other.

2. PROPOSITION. (i) *Let $\mathscr{A}$ be an indexed filter base in X, then $\mathscr{A} \sim \boldsymbol{B}(\boldsymbol{N}(\mathscr{A}))$.*

(ii) *Let α be a net in X, then $\alpha \sim \boldsymbol{N}(\boldsymbol{B}(\alpha))$.*

Proof. (i) Let $\mathscr{A}\colon A \to X$, then the net $\beta = \boldsymbol{N}(\mathscr{A})$ is a mapping of a directed set of pairs (x, a) with $x \in \mathscr{A}(a)$, $a \in A$ and such that $\beta(x, a) = x$. The value of the mapping $\boldsymbol{B}(\beta) = \boldsymbol{B}(\boldsymbol{N}(\mathscr{A}))$ at a point (x, a) is the set $\{y\colon (y, t) \geqslant (x, a)\}$ which can readily be seen to be the set $\mathscr{A}(a)$. Thus, the range of $\boldsymbol{B}(\boldsymbol{N}(\mathscr{A}))$ is the same as the range of $\mathscr{A}$ so that these indexed filter bases are equivalent. (However, the domains of these functions are not the same; indeed, one is an "inflated" version of the other.)

(ii) If $\alpha\colon A \to X$, the value of $\boldsymbol{B}(\alpha)$ at the point a in A is the set $E_a = \{\alpha(t)\colon t \in A,\ a \leqslant t\}$. The domain of $\beta = \boldsymbol{N}(\boldsymbol{B}(\alpha))$ consists of pairs of the form (x, a) with x in E_a and the value of the net β at such a pair is its first coordinate. Now, if $x \in E_a$, then there exists an element t in A with $t \geqslant a$ and $x = \alpha(t)$. Applying the axiom of choice, we infer the existence of a mapping π on such pairs into the set A such that $\pi(x, a) = t \geqslant a$ and $x = \alpha(t)$. Therefore we have

$$\beta(x, a) = x = \alpha(t) = \alpha \circ \pi(x, a).$$

Further, if a is a given element of A, the pair $(\alpha(a), a)$ has the property that when $(\alpha(a), a) \leqslant (x', a')$ (that is, when $a \leqslant a'$), then it follows that $\pi(x', a') \geqslant a' \geqslant a$. This shows that $\beta = \boldsymbol{N}(\boldsymbol{B}(\alpha))$ is a subnet of α.

Conversely, the mapping π' defined on A by $\pi'(a) = (\alpha(a), a)$, can be used to show that α is a subnet of β. This shows that α is equivalent to $\boldsymbol{N}(\boldsymbol{B}(\alpha))$.

The subnet relation. The reader may readily show that if $\mathscr{B}$ is a refinement of $\mathscr{A}$, then it is *not* necessarily the case that $\boldsymbol{N}(\mathscr{B})$ is a subnet of $N(\mathscr{A})$, for the range of $N(\mathscr{B})$ need not be subset of the range of $N(\mathscr{A})$ However, it is the case that there is a net γ which is a subnet of $N(\mathscr{A})$ such that $\mathscr{B} \sim B(\gamma)$. In fact, we can choose γ to be a subnet of $N(\mathscr{B})$.

3. PROPOSITION. *If $\mathscr{B}: B \to X$ is a refinement of $\mathscr{A}: A \to X$, then there is a net γ which is a subnet both of $N(\mathscr{A})$ and $N(\mathscr{B})$ and such that $\mathscr{B} \sim \sim B(\gamma)$.*

Proof. Consider the set C of all triples (x, b, a) where $x \in \mathscr{B}(b) \subseteq \mathscr{A}(a)$ and order C by

$$(x, b, a) \leqslant (x', b', a') \quad \text{if and only if} \quad b \leqslant b' \text{ and } a \leqslant a'.$$

It is easily seen that with this ordering, C is a directed set. If we define γ by $\gamma(x, b, a) = x$, then γ is a net on C to X. Defining π by $\pi(x, b, a) = (x, a)$, it may be proved that γ is a subnet of $N(\mathscr{A})$. In a similar way one shows that γ is a subnet of $N(\mathscr{B})$.

We shall now show that $\mathscr{C} = B(\gamma)$ is equivalent to $\mathscr{B}$. In fact, a typical element of $\mathscr{C}$ is specified by (x, b, a) with $x \in \mathscr{B}(b) \subseteq \mathscr{A}(a)$ and is the set

$$\{x': (x, b, a) \leqslant (x', b', a')\}.$$

It is readily verified that this set coincides with $\mathscr{B}(b)$. In other words, $\mathscr{C}$ is the collection of all sets in $\mathscr{B}$, it is trivial that $\mathscr{C} \leqslant \mathscr{B}$. Conversely, let A_0 be a fixed set in $\mathscr{A}$: since $\mathscr{A} \leqslant \mathscr{B}$, there is a set B_0 in $\mathscr{B}$ such that $B_0 \subseteq A_0$. Now let $B \in \mathscr{B}$; since $\mathscr{B}$ is a filter base there is an element B_1 of $\mathscr{B}$ contained in $B \cap B_0 \subseteq A_0$. Therefore B_1 belongs to $\mathscr{C}$, showing that $\mathscr{C}$ is a refinement of $\mathscr{B}$ and completing the proof.

We now present a substitute for Proposition 2.5 of [1].

4. PROPOSITION. *Let $a: A \to X$ be a net and let $\mathscr{B}$ be a refinement of $B(a)$. Then there exists a subnet β of a such that $B \sim B(\beta)$.*

Proof. By Proposition 3, there is a subnet β of $N(B(a))$ such that $\mathscr{B} \sim B(\beta)$. According to Proposition 2, $a \sim N(B(a))$, so that β is also a subnet of a.

The next result is a partial dual to Proposition 1 (i) and follows directly from Proposition 3.

5. PROPOSITION. *Let $\mathscr{A}, \mathscr{B}$ be indexed filter bases in X. If $\mathscr{A} \leqslant \mathscr{B}$, then there is an indexed filter base $\mathscr{B}'$ which is equivalent to $\mathscr{B}$ and such that $N(\mathscr{A}) \leqslant N(\mathscr{B}')$.*

Let $\mathscr{A}$ and $\mathscr{B}$ be indexed filter bases which are *compositive* (cf. [6], p. 336) in the sense that if $a \in A$, $b \in B$, then $\mathscr{A}(a) \cap \mathscr{B}(b) \neq \emptyset$. Then order $C = A \times B$ by $(a, b) \leqslant (a', b')$ if and only if $a \leqslant a'$ and $b \leqslant b'$, so that C is a directed set and the map $\mathscr{C}: C \to X$ defined by $\mathscr{C}(a, b) = \mathscr{A}(a) \cap \mathscr{B}(b)$ is an indexed filter base. It is quite clear that $\mathscr{A} \leqslant \mathscr{C}$ and $\mathscr{B} \leqslant \mathscr{C}$. It follows from Proposition 4 that if a is a net and $\mathscr{B}$ is an idexed filter base which is compositive with $\mathscr{A} = B(a)$ in the sense defined above, then there is a subnet γ of a such that $B(\gamma)$ is equivalent to the indexed filter base $\mathscr{C}$ constructed above. This construction is Smiley's version of the fundamental lemma on subnets, due to J. L. Kelley [4], p. 278.

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