PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 2

ON DENSE SUBSETS OF BOOLEAN ALGEBRAS

BY

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A set $\mathfrak{S}$ of non-zero elements of a Boolean algebra $\mathfrak{A}$ is said to be *dense* provided for every non-zero element $A \in \mathfrak{A}$ there is an element $B \in \mathfrak{S}$ such that $B \subset A$.

Any dense subset $\mathfrak{S}$ of a Boolean algebra $\mathfrak{A}$ is partially ordered by the Boolean inclusion $\subset$ and satisfies the following disjunctive condition:

(d) if $A, B \in \mathfrak{S}$, and $A \not\subset B$, then there is an element $C \in \mathfrak{S}$ such that $C \subset A$, and no element $D \in \mathfrak{S}$ satisfies simultaneously $D \subset C$ and $D \subset B$.

In other words, if $A, B \in \mathfrak{S}$, and A is not a subelement of B, then A contains a subelement C disjoint with B.

Büchi [1] proved that the necessary condition (d) is also sufficient for $\mathfrak{S}$ to be a dense subset of a Boolean algebra, i. e. the following theorem holds:

(B) *If a set $\mathfrak{S}$ partially ordered by the relation $\subset$ satisfies the disjunctive condition* (d), *then there is a Boolean algebra $\mathfrak{A}$ such that $\mathfrak{S}$ is a dense subset of $\mathfrak{A}$ and the Boolean inclusion in $\mathfrak{A}$ is an extension of $\subset$, i. e., for any elements $A, B \in \mathfrak{S}$, $A \subset B$ if and only if A is a subelement of B in the Boolean algebra $\mathfrak{A}$.*

The Büchi theorem has important applications. For instance, Christensen and Pierce [2] used it to prove the existence of free minimal $\mathfrak{m}$-algebras, $\mathfrak{m}$ being any infinite cardinal. For other applications, see Pierce [3].

In the original formulation of the Büchi theorem the Boolean algebra $\mathfrak{A}$ was additionally required to be complete. Under this additional condition, the complete Boolean algebra $\mathfrak{A}$ is uniquely determined by (B), up to isomorphisms. In practice it is sufficient to prove only (B), because the original stronger version of the Büchi theorem can be obtained from (B) directly: it suffices to replace $\mathfrak{A}$ by its minimal extension (see e. g. Sikorski [4], § 35).

The Büchi proof of (B) was based on the cut method. The purpose of this paper is to present another proof of (B). More precisely, the following theorem will be proved:

(B′) *If a set* $\mathfrak{S}$ *partially ordered by the relation* $\subset$ *satisfies the disjunctive condition* (d), *then there is a field* $\mathfrak{A}$ *of subsets of a space* X *and a mapping* h *from* $\mathfrak{S}$ *into* $\mathfrak{A}$ *such that*

(a) *the class of all sets* $h(A)$, $A \in \mathfrak{S}$, *is dense in the Boolean algebra* $\mathfrak{A}$,

(b) *for any elements* $A, B \in \mathfrak{S}$, $A \subset B$ *if and only if* $h(A)$ *is a subset of* $h(B)$,

(c) *the class of all sets* $h(A)$, $A \in \mathfrak{S}$, *generates* $\mathfrak{A}$.

In the sequel the signs $\wedge$, $-$, 0 will not have the usual lattice-theoretical meaning. They are used only to abbreviate some long sentences. However, the notation applied will show plainly the Boolean content of those sentences.

Two elements $A, B \in \mathfrak{S}$ are said to be *disjoint* if no element $C \in \mathfrak{S}$ satisfies simultaneously $C \subset A$ and $C \subset B$. We shall write then

$$A \wedge B = 0 \quad \text{or} \quad A \subset -B \quad \text{or} \quad B \subset -A.$$

$A \not\subset -B$ will denote the negation of $A \subset -B$.

It follows directly from the adopted notation that

(i) *No element* $A \in \mathfrak{S}$ *satisfies simultaneously* $A \subset B$ *and* $A \subset -B$.

In all expressions below, $+1 \cdot A$ and $-1 \cdot A$ will often replace A and $-A$. By definition,

(ii) *If* $C \subset A$ *and* $A \subset \varepsilon \cdot B$, $\varepsilon = 1$ *or* -1, *then* $C \subset \varepsilon \cdot B$.

For any elements $A_i \in \mathfrak{S}$, and any numbers $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, we shall write

$$\varepsilon_1 \cdot A_1 \wedge \ldots \wedge \varepsilon_n \cdot A_n \neq 0$$

if there exists an element $B \in \mathfrak{S}$ such that $B \in \varepsilon_i \cdot A_i$ for $i = 1, \ldots, n$. In the opposite case we shall write

(1) $$\varepsilon_1 \cdot A_1 \wedge \ldots \wedge \varepsilon_n \cdot A_n = 0.$$

In the case where $n = 2$ and $\varepsilon_1 = \varepsilon_2 = 1$, (1) coincides with the previous definition of $A_1 \wedge A_2 = 0$.

Let H denote the set composed only of the numbers 1 and -1.

(iii) *If* $\mathfrak{S}_0 \subset \mathfrak{S}$, $B \in \mathfrak{S} - \mathfrak{S}_0$, *and* $f \in H^{\mathfrak{S}_0}$ *is a function such that*

(2) $$f(A_1) \cdot A_1 \wedge \ldots \wedge f(A_n) \cdot A_n \neq 0$$

for all elements $A_1, \ldots, A_n \in \mathfrak{S}_0$ $(n = 1, 2, \ldots)$, *then it is possible to define the number* $f(B) \in H$ *in such a way that also*

(3) $$f(A_1) \cdot A_1 \wedge \ldots \wedge f(A_n) \cdot A_n \wedge f(B) \cdot B \neq 0$$

for all elements $A_1, \ldots, A_n \in \mathfrak{S}_0$.

If $f(A_1) \cdot A_1 \wedge \ldots \wedge f(A_n) \cdot A_n \wedge -B \neq 0$ for all $A_1, \ldots, A_n \in \mathfrak{S}_0$, then defining $f(B) = -1$ we get the required extension of f.

Suppose now that there are some elements $B_1, \ldots, B_m \in \mathfrak{S}_0$ such that

(4) $$f(B_1) \cdot B_1 \wedge \ldots \wedge f(B_m) \cdot B_m \wedge -B = 0.$$

Then defining $f(B) = 1$ we obtain the required extension of f. In fact, let $A_1, \ldots, A_n \in \mathfrak{S}_0$. By hypothesis (2),

$$f(A_1) \cdot A_1 \wedge \ldots \wedge f(A_n) \cdot A_n \wedge f \cdot (B_1) \cdot B_1 \wedge \ldots \wedge f(B_m) \cdot B_m \neq 0$$

i. e. there exists an element $C \in \mathfrak{S}$ such that

$$C \subset f(A_i) \cdot A_i \quad \text{for} \quad i = 1, \ldots, n,$$

$$C \subset f(B_j) \cdot B_j \quad \text{for} \quad j = 1, \ldots, m.$$

By (4), $C \not\subset -B$, i. e. the elements C and B are not disjoint. Thus there exists an element $D \in \mathfrak{S}$ such that $D \subset C$ and $D \subset B$. Consequently (see (ii))

$$D \subset f(A_i) \cdot A_i \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad D \subset f(B) \cdot B,$$

which proves (3).

Now let X be the set of all functions $f \in H^{\mathfrak{S}}$ such that (2) holds for all $A_1, \ldots, A_n \in \mathfrak{S}$.

(iv) *For every element* $A \in \mathfrak{S}$ *there exists an* $f \in X$ *such that* $f(A) = 1$.

(iv) follows directly from (iii) by the Kuratowski-Zorn lemma or by the well-ordering principle.

(v) *If* $f \in X$, $f(A) = 1$ *and* $A \subset \varepsilon \cdot B$ $(\varepsilon = \pm 1)$, *then* $f(B) = \varepsilon$.

The hypothesis $f(B) = -\varepsilon$ would imply by (2) that $A \frown -\varepsilon \cdot B \neq 0$, i. e. there would exists an element $C \in \mathfrak{S}$ such that $C \subset A \subset \varepsilon \cdot B$ and $C \subset -\varepsilon \cdot B$. This is impossible (see (i)).

For every element $A \in \mathfrak{S}$, let $h(A)$ denote the set of all $f \in X$ such that $f(A) = 1$.

(vi) *If* $A \subset B$, *then* $h(A)$ *is a subset of* $h(B)$.

This follows directly from (v) (the case of $\varepsilon = 1$).

(vii) *If* A *and* B *are disjoint, then the sets* $h(A)$ *and* $h(B)$ *are disjoint.*

This follows directly from (v) (the case of $\varepsilon = -1$).

(viii) *If* $A \not\subset B$, *then* $h(A)$ *is not a subset of* $h(B)$.

By the disjunctive condition (d), there is a $C \in \mathfrak{S}$ such that $C \subset A$ and $C \frown B = 0$. By (vi), $h(C)$ is a subset of $h(A)$. By (vii), the sets $h(C)$ and $h(B)$ are disjoint. By (iv), the set $h(C)$ is not empty. Thus $h(A)$ is not a subset of $h(B)$.

For any subset Y of X we shall assume the notation

$$+1\cdot Y = Y, \quad -1\cdot Y = -Y = X-Y.$$

(ix) *If for some elements $A_i \in \mathfrak{S}$ and numbers $\varepsilon_i = \pm 1$ the intersection*

$$\varepsilon_1\cdot h(A_1) \cap \ldots \cap \varepsilon_n\cdot h(A_n) \tag{5}$$

is a non-empty set, then there exists an element $C \in \mathfrak{S}$ such that $h(C)$ is a subset of (5).

Since the set (5) is not empty, there exists a function $f_0 \in X$ such that

$$f_0(A_i) = \varepsilon_i \quad \text{for} \quad i = 1, \ldots, n.$$

By (2),

$$\varepsilon_1\cdot A_1 \wedge \ldots \wedge \varepsilon_n\cdot A_n \neq 0,$$

i. e. there exists an element $C \in \mathfrak{S}$ such that $C \subset \varepsilon_i\cdot A_i$ for $i = 1, \ldots, n$. If $f \in h(C)$, i. e. $f(C) = 1$, then $f(A_i) = \varepsilon_i$ by (v), i.e. f belongs to (5). This proves that $h(C)$ is a subset of (5).

To prove theorem (B′), let as assume that $\mathfrak{A}$ is the field (of subsets of X) generated by all the sets $h(A)$, $A \in \mathfrak{S}$. Thus (c) follows directly from the definition of $\mathfrak{A}$. The field $\mathfrak{A}$ is the class of all finite unions of intersections of the form (5). This, by (ix), proves (a). Property (b) follows directly from (vi) and (viii).

Note that in the case where $\mathfrak{S}$ is a dense subset of a given Boolean algebra, the above proof yields the Stone representation theorem. Incidentally it shows also that the Stone space X is a subset of the Cantor space $H^{\mathfrak{S}}$.

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Reçu par la Rédaction le 27. 8. 1962

COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 2

AXIOMS AND SOME PROPERTIES OF POST ALGEBRAS

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Introduction. The notion of n-valued logic was introduced first by E. L. Post, [5], in 1921. A special case of this notion, the 3-valued logic, was formulated earlier by J. Łukasiewicz, [3], in 1920.

It is well known that there is a Boolean algebra corresponding to the two-valued logic (see, e. g. S. Mazurkiewicz [4], p. 55). P. C. Rosenbloom [6] published in 1942 the first system of axioms of the algebra corresponding to the n-valued logic of E. L. Post. He has called this algebra *Post algebra*. However, Rosenbloom's system of axioms was a very difficult one.

G. Epstein [1] was the first who simplified this theory by making use of the existence of a Boolean algebra underlying a given Post algebra.

P. C. Rosenbloom has already noticed that the theory of the Post algebra may be applied in other branches of mathematics, not only in logic.

The purpose of the present paper is to give a few simple systems of axioms of the Post algebra and to formulate some of its properties, similar to those of a Boolean algebra.

In section 1 a distributive lattice called P_0-lattice is examined, some properties of which give us a good position to formulate in section 3 a few simple systems of axioms of the Post algebra. Section 2 contains Epstein's difinition of a Post algebra and some lemmas rewritten from Epstein paper [1]. Section 4 contains some simple lemmas on the extension of Boolean homomorphisms to Post homomorphisms and some properties of m-valued Post homomorphisms. In section 5 a normed measure on a given Post algebra is defined. Section 6 contains a set-theoretical representation of a Post algebra. In section 7 a congruence relation is defined which makes it possible to obtain a Post algebra from a P_0-lattice.

The most essential results of the present paper were published earlier, [8], without proofs.

I should like to remark, finally, that the paper is almost self-contained; I have only used one or two results of other authors.