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A REMARK ON A PROBLEM OF M. KRZYŻAŃSKI CONCERNING SECOND ORDER PARABOLIC EQUATIONS

BY

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Consider the linear second order equation of parabolic type

$$Fu \equiv \sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = 0, \tag{1}$$

where $x = (x_1, \ldots, x_m)$ varies in the whole m-dimensional Euclidean space E^m and $0 < t < T$. Denote by D^T the topological product of E^m with the interval $(0, T)$.

By a solution of (1) is meant a function $u(x,t)$, which is continuous in the closure $\overline{D^T}$ of D^T and which has continuous partial derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$, $\partial u/\partial t$ in D^T satisfying (1).

Let $u(x,t)$ be a solution of (1) satisfying the initial condition

$$u(x,0) = \varphi(x) \quad \text{for} \quad x \in E^m, \tag{2}$$

$\varphi(x)$ being a given continuous function. Assume that there exist positive constants M, K such that the solution fulfils the inequality

$$u(x,t) \geqslant -M\exp(K|x|^2) \quad \text{for} \quad (x,t) \in D^T, \tag{3}$$

where $|x| = (\sum_{i=1}^{m} x_i^2)^{1/2}$. Krzyżański's problem is whether the condition (3) is sufficient for equation (1) to have at most one solution satisfying (2). It is known that, for instance, the condition

$$|u(x,t)| \leqslant M\exp(K|x|^2), \quad (x,t) \in D^T, \tag{4}$$

is sufficient if certain growth conditions concerning the coefficients are fulfilled (see [2]).

In the case when $\varphi(x) \equiv 0$ a positive answer can be obtained from the below mentioned theorems. Using the fundamental solution constructed by Dressel, Friedman [1] has proved the following

THEOREM I. *If the following conditions hold:*

1° *there is a positive constant L such that for all $(x,t)\in D^T$ and for every real vector $(\xi_1, \ldots, \xi_m)$*

$$\sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \geqslant L\sum_{i=1}^{m}\xi_i^2;$$

2° *the functions*

$$a_{ij},\ \frac{\partial a_{ij}}{\partial x_\lambda},\ \frac{\partial^2 a_{ij}}{\partial x_\lambda \partial x_\mu},\ \frac{\partial a_{ij}}{\partial t},\ b_i,\ \frac{\partial b_i}{\partial x_\lambda},\ c$$

are locally Hölder-continuous and bounded in $\overline{D^T}$;

3° *the solution $u(x,t)$ of* (1) *is non-negative in the strip D^T, then the condition*

(5) $$u(x,0) = 0 \quad \text{for} \quad x\in E^m$$

yields $u(x,t)\equiv 0$ in D^T.

The next theorem constitutes a certain generalization of a theorem established by Krzyżański [2].

THEOREM II. *If the following conditions hold:*

1° *the quadratic form*

$$\sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j$$

is positive definite for $(x,t)\in D^T$,

2° *there exist positive constants $A_0, \ldots, A_4$, such that*

$$|a_{ij}(x,t)| \leqslant A_0, \quad |b_j(x,t)| \leqslant A_1|x|+A_2, \quad c(x,t)\leqslant A_3|x|^2+A_4,$$

3° *the condition* (3) *is satisfied,*

then the inequality $u(x,0)\geqslant 0$, $x\in E^m$, implies $u(x,t)\geqslant 0$ in $\overline{D^T}$ (¹).

The proof of this theorem is a slight modification of the proof of the theorem of M. Krzyżański. Namely, we make use of some properties of the function

$$H(x,t;k) = \exp\left\{\frac{k|x|^2}{1-\mu(k)t}+\nu(k)t\right\}, \qquad k>K.$$

In the papers [2], [3] it has been shown that, for a given k, the constants $\mu(k)$, $\nu(k)$ appearing in this function can be chosen so that the relation $FH \leqslant -NH$ holds for every point (x,t) contained in a sufficiently narrow strip D^h $(h<1/\mu(k))$, N being an arbitrary positive

(¹) In the paper of M. Krzyżański instead of (3) the stronger condition (4) is assumed.

number. It is sufficient to prove the theorem for the domain D^h, because if $T>h$, then a well known change of the variable t permits to prove it for the whole strip D^T.

The substitution

(6) $$u(x,t) = v(x,t)H(x,t;k), \qquad (x,t)\in D^h,$$

transforms (1) into

(7) $$\sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 v}{\partial x_i\partial x_j} + \sum_{j=1}^{m}\bar{b}_j(x,t)\frac{\partial v}{\partial x_j} + \bar{c}(x,t)v - \frac{\partial v}{\partial t} = 0,$$

with $\bar{c}(x,t) = FH/H \leqslant -N < 0$.

Now, let $(\bar{x},\bar{t})$ be an arbitrary fixed point contained in $\overline{D^h}$. We shall show that $u(\bar{x},\bar{t})\geqslant 0$. For this purpose let us choose an arbitrary increasing sequence $\{R_n\}$, $R_n>|\bar{x}|$, $R_n\to\infty$ as $n\to\infty$. By D_n^h we denote the part of D^h lying inside the cylindrical surface Γ_n with the equation $|x| = R_n$, whereas by σ_n^h we denote the part of the surface Γ_n contained in D^h. By virtue of assumption 3° and transformation (6) we obtain, for $(x,t)\in\sigma_n^h$,

$$v(x,t) \geqslant -\frac{M\exp(K|x|^2)}{\exp\left\{\dfrac{k|x|^2}{1-\mu(k)t}+\nu(k)t\right\}} = -\frac{M\exp(KR_n^2)}{\exp\left\{\dfrac{kR_n^2}{1-\mu(k)t}+\nu(k)t\right\}}.$$

Hence it follows that for every $\varepsilon>0$ there exists an integer $n_0(\varepsilon)$ such that for $n>n_0$ the inequality

(8) $$v(x,t) > -\varepsilon$$

holds for $(x,t)\in\sigma_n^h$. This inequality, by our assumption, is also satisfied on the part of the boundary of D_n^h lying on the hyperplane $t=0$. Therefore, by the principle of minimum, the inequality (8) is satisfied for $(x,t)\in\overline{D_n^h}$. In particular we have $v(\bar{x},\bar{t})>-\varepsilon$, for every $\varepsilon>0$. This means that $v(\bar{x},\bar{t})\geqslant 0$, and thereby $u(\bar{x},\bar{t})\geqslant 0$, q. e. d.

From theorem II it follows that the theorem I will be true if instead of the assumption that $u(x,t)\geqslant 0$, the condition (3) is adopted. This gives a positive solution of Krzyżański's problem in the particular case when $\varphi(x)\equiv 0$ and under the assumptions of theorem I concerning the coefficients.

Note that if a function $u(x,t)$ fulfils equation (1), then so does the function $\bar{u}(x,t) \overset{\text{df}}{=} -u(x,t)$. If, now, the assumptions 1° and 2° of theorem I are satisfied, then condition (5) and

$$u(x,t)\leqslant M\exp(K|x|^2), \qquad (x,t)\in D^T,$$

imply $u(x,t)\equiv 0$ in the strip D^T.

Friedman [1] has proved that if there exists a constant $K_0 > 0$ such that

$$\int_0^T \int_{E^m} \exp(-K_0|x|^2)|u(x,t)|\,dt\,dx < +\infty$$

and if the initial condition (5) is satisfied, then $u(x,t) \equiv 0$ (under the assumptions 1° and 2° of theorem I concerning the coefficients). Write

$$u^+(x,t) = \begin{cases} u(x,t), & \text{if} \quad u(x,t) \geqslant 0, \\ 0, & \text{if} \quad u(x,t) < 0. \end{cases}$$

There exists a supposition that the condition

$$\int_0^T \int_{E^m} \exp(-K_0|x|^2)u^+(x,t)\,dt\,dx < +\infty$$

is sufficient in order to make the solution $u(x,t)$ of equation (1), satisfying (5), vanish identically in D^T.

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EVALUATIONS OF SOLUTIONS OF A SECOND ORDER PARABOLIC EQUATION

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Let us consider the equation

$$(1) \qquad \Delta u - \frac{\partial u}{\partial t} + c(x,t)u = 0, \qquad \text{where} \qquad \Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i}, \; x = (x_1, \ldots, x_m).$$

The following theorem has been established by Krzyżański [4]:

THEOREM K. *Assume the coefficient $c(x,t)$ to be defined and continuous when x varies in the m-dimensional Euclidean space E^m, $t > 0$, and to satisfy the Lipschitz's condition with respect to x. Suppose there exist constants α, β, A, B; $\alpha > 0$, $A > 0$, $B > 0$, such that $\alpha^2|x|^2 + \beta \leqslant c(x,t) \leqslant A|x|^2 + B$ for $x \in E^m$, $t > 0$, where $|x| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}$. If a solution $u(x,t)$ of equation (1) satisfies the condition $u(x,0) \geqslant N > 0$ for $x \in E^m$ and belongs to the so-called class E_2, then*

$$u(x,t) \geqslant M\exp(K|x|^2 \tan 2\alpha t) \quad \text{for} \quad x \in E^m, \; t \in \left(0, \frac{\pi}{4\alpha}\right),$$

M, K, being positive constants.

In the proof the author mentioned above has applied a fundamental solution, constructed in [7], which requires certain assumptions concerning a regularity of coefficients.

In this note we prove similar theorems for a more general equation of the form

$$(2) \qquad Fu \equiv \sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t)$$

by means of a method which does not use the fundamental solution. A quasi-linear equation will also be discussed.

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