Expressing (6) in terms of (7) and adding (5) and (6) we obtain the equations of an arc of Γ as given in (1). Since τ in (7) is a strictly monotonic function of t, the tangent of Γ coinciding with the tangent of Γ_2 varies in a monotonic manner and therefore Γ is convex.

The extremal functions corresponding to the boundary Γ have the form (1) with suitably chosen η, ε ($|\eta| = |\varepsilon| = 1$). For $t = (1-r^2)^{-1/2}$ we have $\tau = 1$, and then the imaginary part of w on Γ attains its maximal value

$$\arccos\frac{1-r^2}{1+r^2} + 2\arccos\sqrt{1-r^2} = 2(\arctan r + \arcsin r)$$

and this gives the Theorem 2. The bound is sharp and is attained for functions of the form (1).

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INTEGRATION OF THE FIRST ORDER PARTIAL DIFFERENTIAL INEQUALITY WITH DISTRIBUTIONS

BY

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In the present paper we are interested in the integration of the inequality

$$\frac{\partial u}{\partial t} \leqslant \sum_{j|1}^{n} a_j(t)\frac{\partial u}{\partial x_j} + b(t)u$$

where u is a distribution on a suitably chosen space. We use in our considerations a Fourier transform which maps the above inequality into a simpler one which can be easily solved.

1. In notations and formulas we follow here [1] and [2]. By K we denote the space of complex-valued functions $\varphi(x_1, \ldots, x_n)$ of class C^∞ and compact supports, with the usual topology. The Fourier image of K is the space Z: the elements of Z are defined by

$$\hat{\varphi}(\sigma) = \int_{R^n} \varphi(x) e^{i(x,\sigma)} dx = F(\varphi) \tag{1}$$

with $\varphi \epsilon K$. As usually $(x, \sigma) = \sum_{i|1}^{n} x_i\sigma_i$, σ_i real, $x = (x_1, \ldots, x_n)$.

The Fourier transform of a distribution $f \epsilon K'$ is defined by formula

$$(\hat{f}, F(\varphi)) = (2\pi)^n (f, \varphi).$$

Moreover, for differentiation we have

$$\left(\widehat{\frac{\partial f}{\partial x_j}}, \hat{\varphi}\right) = ((-i\sigma_j)\hat{f}, \hat{\varphi})$$

or briefly

$$\widehat{\frac{\partial f}{\partial x_j}} = (-i\sigma_j)\hat{f}. \tag{2}$$

Let $\varphi \in Z$. A function $a(\sigma)$ is a multiplier in Z if $a\hat{\varphi} \in Z$ for $\hat{\varphi} \in Z$.

Let $a(\sigma)$ be a multiplier. The product $\bar{a}g$, where $g \in Z'$ is defined by formula

$$(3)\qquad (\bar{a}g, \hat{\varphi}) = (g, a\hat{\varphi}).$$

In this sense $a = -i\sigma_j$ is a multiplier in formula (2). Remark now that every function $a(i\sigma)$ (see [1], p. 128-130) which is analytic and satisfies

$$|a(i\sigma)| \leqslant C(1+|\sigma^m|)e^{b|\sigma|}$$

for some $m \geqslant 0$, $b \geqslant 0$ is a multiplier in Z. Take now the function

$$(4)\qquad a(\sigma) = \exp\Big\{i\sum_{j|1}^{n}\gamma_j\sigma_j+\gamma\Big\}$$

with real γ_j, γ. We have $|a(\sigma)| = e^{\gamma}$.

Hence, every function of form (4) is a multiplier in Z.

We now briefly discuss positive distributions. First the definition: the distribution $f \in K'$ is called *positive*, $f \geqslant 0$, if $(f, \varphi) \geqslant 0$ for $\varphi \geqslant 0$, $\varphi \in K$. It is known (see [2]) that a necessary and sufficient condition for f to be positive is that $(f, \varphi\bar{\varphi}) \geqslant 0$ for every $\varphi \in K$. Using the Fourier transform we can say that $f \geqslant 0$ is equivalent to $\big(\hat{f}, \int|\varphi(u)|^2 e^{i(\sigma,u)}du\big) \geqslant 0$ for every $\varphi \in K$ (for details see [2], p. 207-208). We write then $\hat{f} \geqslant 0$.

2. We can now prove the following lemma:

LEMMA. *Suppose that $f \geqslant 0$. Then $G = \hat{f}\exp\big(-i\sum_{j|1}^{n}\gamma_j\sigma_j+\gamma\big) \geqslant 0$ for real γ_j, γ.*

Proof. The function $\exp\big(i\sum_{j|1}^{n}\gamma_j\sigma_j+\gamma\big)$ is a multiplier in Z. Hence the distribution

$$G = \hat{f}\exp\Big(-i\sum_{j|1}^{n}\gamma_j\sigma_j+\gamma\Big)$$

is well defined. In order to show that $G \geqslant 0$ one must show that

$$\varDelta = \Big(G, \int|\varphi(u)|^2e^{i(\sigma,u)}du\Big) \geqslant 0, \qquad \varphi \in K.$$

We have by formula (3)

$$\varDelta = \Big(G, \int|\varphi(u)|^2e^{i(\sigma,u)}du\Big) = \Big[\hat{f}\exp\Big(-i\sum_{j|1}^{n}\gamma_j\sigma_j+\gamma\Big), \int|\varphi(u)|^2e^{i(\sigma,u)}du\Big]$$

$$= \Big[\hat{f}, \int|\varphi(u)|^2e^{i(\sigma,u)}du\cdot\exp\Big(i\sum_{j|1}^{n}\gamma_j\sigma_j+\gamma\Big)\Big].$$

Hence

$$\varDelta = \Big[\hat{f}, e^{\gamma}\int|\varphi(u_1, \ldots, u_n)|^2\exp\Big(i\sum_{j|1}^{n}(u_j+\gamma_j)\sigma_j\Big)du_1\ldots du_n\Big].$$

But

$$e^{\gamma}\int|\varphi(u_1, \ldots, u_n)|^2\exp\Big(i\sum_{j|1}^{n}(u_j+\gamma_j)\sigma_j\Big)du_1\ldots du_n$$

$$= e^{\gamma}\int|\varphi(\tau_1-\gamma_1, \ldots, \tau_n-\gamma_n)|^2\exp\Big(i\sum_{j|1}^{n}\tau_j\sigma_j\Big)d\tau_1\ldots d\tau_n,$$

and consequently

$$\varDelta = \Big(\hat{f}, \int|\psi(u)|^2e^{i(\sigma,u)}du\Big), \qquad \hat{\gamma} = (\gamma_1, \ldots, \gamma_n),$$

where $\psi(u) = e^{\gamma/2}\varphi(u-\hat{\gamma})$. But $\hat{f} \geqslant 0$. Hence $\varDelta \geqslant 0$, q. e. d.

In the sequel we write $f \leqslant g$ if $0 \leqslant g-f$ and $f, g \in K'$.

The spatial derivatives are taken in distribution sense. We shall prove the following theorem:

THEOREM. *Let the real valued functions $a_j(t)$, $j = 1, \ldots, n$, and $b(t)$ be continuous for $t \geqslant 0$. Suppose we are given a distribution-valued function $u(t) \in K'$, weakly differentiable in t, which satisfies*

$$(5)\qquad \frac{\partial u}{\partial t} \leqslant \sum_{j|1}^{n}a_j(t)\frac{\partial u}{\partial x_j}+b(t)u$$

for $t \geqslant 0$ and

$$(6)\qquad u(0) \leqslant 0.$$

Under our assumptions the inequality $u(t) \leqslant 0$ holds for $t \geqslant 0$.

Proof. Let

$$(7)\qquad R(t) = \frac{\partial u}{\partial t}-\sum_{j|1}^{n}a_j(t)\frac{\partial u}{\partial x_j}-b(t)u,$$

$R(t)$ is a negative distribution. By (2) and (7) we have

$$\frac{d\hat{u}}{dt} = \sum_{j|1}^{n}(-i\sigma_j)a_j(t)\hat{u}(t)+b(t)\hat{u}(t)+\hat{R}(t)$$

$$= \Big\{i\sum_{j|1}^{n}\big(-\sigma_ja_j(t)\big)+b(t)\Big\}\hat{u}(t)+\hat{R}(t).$$

Hence

$$(8)\qquad \hat{u}(t) = \exp\Big\{\int_0^t \Big[i\sum_{j|1}^{n}\big(-\sigma_j a_j(\tau)\big)+b(\tau)\Big]d\tau\Big\}\hat{u}(0)+$$
$$+\int_0^t \hat{R}(\eta)\exp\Big\{\int_\eta^t \Big[i\sum_{j|1}^{n}\big(-\sigma_j a_j(\tau)\big)+b(\tau)\Big]d\tau\Big\}d\eta.$$

On the other hand, $\hat{u}(0)\leqslant 0$ and $\hat{R}(\eta)\leqslant 0$, and the function $\xi(t,\eta;\sigma) = \exp\{\int_\eta^t [i\sum_{j|1}^{n}(-\sigma_j a_j(\tau))+b(\tau)]d\tau\}$ is a multiplier in Z for fixed t and η. Hence, by our lemma, $\xi(t,\eta;\sigma)\hat{u}(0)\leqslant 0$ and $\xi(t,\eta;\sigma)\hat{R}(\eta)\leqslant 0$ for $\eta\leqslant t$. Obviously $\int_0^t \xi(t,\eta;\sigma)\hat{R}(\eta)d\eta\leqslant 0$. Hence both parts of the right-hand member of (8) are negative, q. e. d.

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SOME REMARKS ON A CERTAIN METHOD OF SUCCESSIVE APPROXIMATIONS IN DIFFERENTIAL EQUATIONS

BY

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In papers [1] and [2] a method of successive approximations in differential equations was discussed. Some sufficient conditions for the convergence of iteration process were given. These conditions were obtained by reducing the problem to the solving of a system of Volterra's equations by successive approximations method. In the present paper we shall give some remarks which allow to weaken the assumption of theorems formulated in [1] and [2].

1. Let us consider Volterra's integral equation of the form

$$(1)\qquad x(t) = A(t)x(t)+\int_0^t B(t,\xi)x(\xi)d\xi+f(t),$$

where matrices $A(t)$, $B(t,\xi)$ are continuous for $t\geqslant 0$, and $t\geqslant 0$, $0\leqslant\xi<t$ respectively; vector function $f(t)$ is continuous for $t\geqslant 0$.

Definition 1. Let $\|x\|$ be an arbitrary norm of the vector

$$x = \begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

i. e. a non-negative number satisfying conditions:

a) $\|x\|>0$, for $x\neq 0$ and $\|0\|=0$,

b) $\|cx\| = |c|\cdot\|x\|$, c — an arbitrary real number,

c) $\|x+y\|\leqslant\|x\|+\|y\|$.

Definition 2. Let $\|A\| = \max_{\|x\|=1}\|Ax\|$ be the norm of matrix A (see [3], p. 124-127).