great $|p_k/q_k|$ we have $|P(p_k/q_k)| \geqslant M_3|p_k/q_k|^m$ with a suitable positive M_3. It follows that the left side of (3) is at least

$$M_3 \cdot \frac{q_k^m}{|p_k|} \cdot \frac{|p_k|^m}{q_k^m} = M_3|p_k|^{m-1},$$

but the right side of (3) is bounded, and so we infer that $m = 0, 1$ and a fortiori $n = 0, 1$, which means that $F(t)$ is a homography. In the case $n > m$ the proof is almost the same, as can be easily seen from the symmetry of (1). We proved thus that if $x \neq \infty$, $F(x) \neq 0$, ∞ and $s(x)$ is sufficiently great, then $s(F(x)) > s(x)$. But in all remaining cases $s(x)$ is bounded by a constant. Consequently if $F(t)$ is not a homography the condition (ii) of lemma 1 is verified, which completes the proof of the theorem.

REFERENCES

[1] W. Narkiewicz, *On polynomial transformations*, Acta Arithmetica 7 (1962), p. 241-249.

[2] — *On polynomial transformations II*, Acta Arithmetica 8 (1962), p. 11-19.

[3] A. Schinzel, *P 417*, this volume, p. 187.

Reçu par la Rédaction le 19. 5. 1962

COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 1

ON THE DERIVATIVE OF CLOSE-TO-CONVEX FUNCTIONS

BY

J. KRZYŻ (LUBLIN)

Let D be a simply connected domain of hyperbolic type, i. e. a domain conformally equivalent to an open circle. Then the following definitions of close-to-convexity of D may be considered.

(B): D is said to be *close-to-convex*, or *accessible from outside along rays* [1], if the complement of D can be represented as a union of closed rays which do not cross each other.

(K): D is said to be *close-to-convex*, if for the function $f(z)$ mapping D conformally onto the unit circle $K = \{z: |z| < 1\}$ a univalent and convex function $\Phi(z)$, $z \in K$, can be chosen so that $\Re\{f'(z)/\Phi'(z)\} > 0$ for any $z \in K$ (see [2]).

As pointed out by Lewandowski [3], both definitions of close-to-convexity are equivalent.

For a domain D bounded by a Jordan curve Γ with a continuously changing tangent another equivalent definition of close-to-convexity was given in [2].

(K_1): D is said to be *close-to-convex*, if the maximal angle of a clockwise rotation of the outward normal along any subarc of Γ described in the positive (counter-clockwise) direction does not surpass π. Therefore we can also consider close-to-convex curves.

In particular, the class (L) of univalent functions $f(z) = z + a_2z^2 + \ldots$ mapping K onto close-to-convex domains, i. e. the class of close-to-convex functions (introduced independently by Biernacki [1] and Kaplan [2]), may be considered. The class (L) contains functions such as convex, starlike, convex in one direction [5], starlike with respect to symmetric points [6], functions with the derivative of positive real part etc.

In [1], which does not seem to be universally known, Biernacki determined the region of variability of the functionals $\{z/f(z)\}$, $\{zf'(z)/f(z)\}$, for a fixed $z \in K$ and f ranging over (L). In this article we solve an analogous problem for $\log f'(z)$ (Theorem 1), and hence we deduce the precise estimates of $\arg f'(z)$ for $f \in (L)$ (Theorem 2). In spite of the fact that the class (L) is rather important, this problem has remained unnoticed to the best of the author's knowledge. We have the following

THEOREM 1. *Let $f(z) = z + a_2 z^2 + \ldots$ be a function regular and close-to-convex in the unit circle K. Then the set of possible values $u+iv$ of $\log f'(re^{i\theta})$ for fixed r and θ $(0 < r < 1,\ \theta$ real$)$ is a convex domain $D(r)$ symmetric with respect to the real axis Ou and with respect to the line $u = \log 1/(1-r^2)$. The boundary of $D(r)$ arises by reflecting the convex arc Γ:*

$$
(1) \qquad \begin{aligned}
u &= \log t \left\{ t^2 - \frac{1}{1-r^2} + \left[\left(t^2 - \frac{1}{1-r^2} \right)^2 + t^2 \right]^{1/2} \right\}, \\
v &= \arccos \frac{1-r^2}{1+r^2} \left\{ 1 + \left[t - \frac{1}{t(1-r^2)} \right]^2 \right\}^{1/2} + \\
&\quad + 2 \arccos \frac{1+t^2(1-r^2)}{2t}, \qquad (1-r^2)^{-1/2} \leqslant t \leqslant (1-r)^{-1},
\end{aligned}
$$

with respect to the axes of symmetry.

The functions corresponding to the boundary of $D(r)$ have the form

$$
(2) \qquad f(z) = \int_0^z \frac{1}{(1-\eta\zeta)^2} \, \frac{1+\varepsilon\zeta}{1-\varepsilon\zeta} \, d\zeta
$$

with suitably chosen ε, η, where $|\varepsilon| = |\eta| = 1$.

As a corollary of Theorem 1, we obtain the

THEOREM 2. *Let $f(z) = z + a_2 z^2 + \ldots$ be a function regular and close-to-convex in the unit circle K. We have*

$$
(3) \qquad |\arg f'(re^{i\theta})| \leqslant 2(\arcsin r + \arctan r) \qquad (0 < r < 1,\ \theta \text{ real})
$$

with the sign of equality only for functions of the form (2) *with suitably chosen ε, η.*

In both theorems we consider principal branches of logarithm and argument changing continuously from the initial values $\log 1 = \arg 1 = 0$.

Proof of Theorem 1. If $f \in (L)$, so is $e^{i\theta} f(ze^{-i\theta})$ with arbitrary real θ. Hence the set of values of $f'(z)$, $f \in (L)$, $z \in K$ being fixed, is identical with the set of values of $f'(ze^{-i\theta})$, and therefore we may confine ourselves to real, positive z. According to the definition (K) and a well-known relation between convex and starlike functions, we have $f(z) = \int_0^z \zeta^{-1} f^*(\zeta) \times$ $\times p(\zeta) d\zeta$, where $f^*(z) = z + A_2 z^2 + \ldots$ is a function regular in K and starlike with respect to the origin, and $p(z) = 1 + c_1 z + \ldots$ is regular and of positive real part in K. Since f^* and p may be chosen independently of each other, we have $\log f'(r) = w_1 + w_2$, where w_1, w_2 range over the sets $D_1(r)$, $D_2(r)$ of variablitily of $\log\big(r^{-1} f^*(r)\big)$ and $\log p(r)$, respectively. Now, according to Marx [4] the variability region of $\big(r^{-1} f^*(r)\big)^{1/2}$ is the closed disc C_1 with the diameter $[(1+r)^{-1}, (1-r)^{-1}]$ and the functions f^* corresponding to the boundary of C_1 have the form $z(1-\eta z)^{-2}$ with $|\eta| = 1$. It is a well-known consequence of Herglotz's structural formula for functions with the positive real part that the domain of variability of $p(r)$ for a fixed r, $0 < r < 1$, is the closed disc C_2 with the diameter $[(1-r)(1+r)^{-1}, (1+r)(1-r)^{-1}]$, and the functions $p(z)$ corresponding to the boundary of C_2 have the form $(1+\varepsilon z)(1-\varepsilon z)^{-1}$ with $|\varepsilon| = 1$. Now, if C is a closed disc in the (z)-plane with the diameter $[a, b]$ $(0 < a < b)$, its image by $\log z$ in the (w)-plane $(w = u+iv)$ is a convex domain with the boundary

$$
(4) \qquad \begin{aligned}
u &= \log t, \\
v &= \mp \arccos \frac{t^2 + ab}{t(a+b)} \qquad (a \leqslant t \leqslant b),
\end{aligned}
$$

symmetric with respect to the real axis and the line $u = \frac{1}{2}\log ab$, since $t_1 t_2 = ab$ implies $\frac{1}{2}(u_1+u_2) = \frac{1}{2}\log ab$, and $v(t_1) = \mp v(t_2)$. Therefore the regions $D_1(r)$, $D_2(r)$ of variability of $\log\big(r^{-1} f^*(r)\big)$ and $\log p(r)$ are convex symmetric domains with boundaries Γ_1, Γ_2:

$$
(5) \qquad \begin{aligned}
u_1 &= 2\log t, \\
v_1 &= \pm 2 \arccos \tfrac{1}{2}\big(t^{-1} + t(1-r^2)\big), \\
&(1+r)^{-1} \leqslant t \leqslant (1-r)^{-1},
\end{aligned}
$$

$$
(6) \qquad \begin{aligned}
u_2 &= \log \tau, \\
v_2 &= \pm \arccos \tfrac{1}{2} \, \frac{1-r^2}{1+r^2} \left(\tau + \frac{1}{\tau} \right), \\
&(1-r)(1+r)^{-1} \leqslant \tau \leqslant (1+r)(1-r)^{-1},
\end{aligned}
$$

Γ_2 being symmetric with respect to both axes Ou, Ov.

Now, $D(r) = \{w \colon w = w_1 + w_2\}$, with $w_1 \in D_1(r)$, $w_2 \in D_2(r)$, and hence $D(r)$ may be obtained as the envelope of $D_2(r)$ whose centre is describing without rotation the curve Γ_1. If w is the point of the envelope such that $w = w_1(t) + w_2(\tau)$, the tangent of the envelope at w, the tangent of Γ_1 at $w_1(t)$, and the tangent of Γ_2 at $w_2(\tau)$, are parallel. This implies the equality

$$
(7) \qquad \tau = t - \frac{1}{t(1-r^2)} + \left\{ 1 + \left[t - \frac{1}{t(1-r^2)} \right]^2 \right\}^{1/2}
$$

for $(1-r^2)^{-1/2} \leqslant t \leqslant (1-r)^{-1}$.

Expressing (6) in terms of (7) and adding (5) and (6) we obtain the equations of an arc of Γ as given in (1). Since τ in (7) is a strictly monotonic function of t, the tangent of Γ coinciding with the tangent of Γ_2 varies in a monotonic manner and therefore Γ is convex.

The extremal functions corresponding to the boundary Γ have the form (1) with suitably chosen η, ε ($|\eta| = |\varepsilon| = 1$). For $t = (1-r^2)^{-1/2}$ we have $\tau = 1$, and then the imaginary part of w on Γ attains its maximal value

$$\arccos\frac{1-r^2}{1+r^2} + 2\arccos\sqrt{1-r^2} = 2(\arctan r + \arcsin r)$$

and this gives the Theorem 2. The bound is sharp and is attained for functions of the form (1).

REFERENCES

[1] M. Biernacki, *Sur la représentation conforme des domaines linéairement accessibles*, Prace Matematyczno-Fizyczne 44 (1936), p. 293-314.

[2] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Mathematical Journal 1 (1952), p. 169-185.

[3] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes, I*, Annales Universitatis Mariae Curie-Skłodowska 12 (1958), p. 131-146; *II*, ibidem 14 (1960), p. 19-46.

[4] A. Marx, *Untersuchungen über schlichte Abbildungen*, Mathematische Annalen 107 (1933), p. 40-67.

[5] M. S. Robertson, *Analytic functions star-like in one direction*, American Journal of Mathematics 58 (1936), p. 465-472.

[6] Kôichi Sakaguchi, *On a certain univalent mapping*, Journal of Mathematical Society of Japan 11 (1959), p. 72-75.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 20. 1. 1962

COLLOQUIUM MATHEMATICUM

VOL. X 1963 FASC. 1

INTEGRATION OF THE FIRST ORDER PARTIAL DIFFERENTIAL INEQUALITY WITH DISTRIBUTIONS

BY

W. MLAK (CRACOW)

In the present paper we are interested in the integration of the inequality

$$\frac{\partial u}{\partial t} \leqslant \sum_{j|1}^{n} a_j(t)\frac{\partial u}{\partial x_j} + b(t)u$$

where u is a distribution on a suitably chosen space. We use in our considerations a Fourier transform which maps the above inequality into a simpler one which can be easily solved.

1. In notations and formulas we follow here [1] and [2]. By K we denote the space of complex-valued functions $\varphi(x_1, \ldots, x_n)$ of class C^∞ and compact supports, with the usual topology. The Fourier image of K is the space Z: the elements of Z are defined by

(1) $$\hat{\varphi}(\sigma) = \int_{R^n} \varphi(x) e^{i(x,\sigma)} dx = F(\varphi)$$

with $\varphi \epsilon K$. As usually $(x, \sigma) = \sum_{i|1}^{n} x_i \sigma_i$, σ_i real, $x = (x_1, \ldots, x_n)$.

The Fourier transform of a distribution $f \epsilon K'$ is defined by formula

$$\big(\hat{f}, F(\varphi)\big) = (2\pi)^n (f, \varphi).$$

Moreover, for differentiation we have

$$\left(\widehat{\frac{\partial f}{\partial x_j}}, \hat{\varphi}\right) = \big((-i\sigma_j)\hat{f}, \hat{\varphi}\big)$$

or briefly

(2) $$\widehat{\frac{\partial f}{\partial x_j}} = (-i\sigma_j)\hat{f}.$$