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MARCZEWSKI INDEPENDENCE IN LATTICES AND SEMILATTICES

BY

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In this paper we shall apply to lattices and semilattices the general notion of independence introduced by Marczewski in [2].

1. Definitions. Following Birkhoff [1], by an *algebra* $\mathscr{A} = (A; \{f_\gamma\}_{\gamma\in\Gamma})$ we mean a set A of *elements* with a class $F = \{f_\gamma\}_{\gamma\in\Gamma}$ of (so called) *fundamental operations*, each f_γ being supposed to be an A-valued function of finite variables defined on A. Further, the class $F^{(n)}$ of *algebraic operations of n variables* on $\mathscr{A}$ is defined as the smallest class of functions satisfying the following two conditions:

(i) $F^{(n)}$ contains all *selector operations* $s_k^{(n)}$ $(k = 1, \ldots, n)$ of n variables defined by the formulas

$$s_k^{(n)}(x_1, \ldots, x_n) = x_k \qquad (x_1, \ldots, x_n \in A);$$

(ii) If $f_1, \ldots, f_r \in F^{(n)}$ and f is a fundamental operation of r variables, then the operation g defined by

$$g(x_1, \ldots, x_n) = f\big(f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n)\big) \qquad (x_1, \ldots, x_n \in A)$$

also belongs to $F^{(n)}$.

The single selector operation of one variable will be called *identity operation* of A and denoted briefly by s instead of $s_1^{(1)}$.

A subset S of A will be called *M-independent* in $\mathscr{A}$ (see [2]) if S has the following property: Given any algebraic operations g and h of n variables on $\mathscr{A}$, if there exist different elements $a_1, \ldots, a_n$ in S such that

$$g(a_1, \ldots, a_n) = h(a_1, \ldots, a_n),$$

then

$$g(x_1, \ldots, x_n) = h(x_1, \ldots, x_n)$$

for each sequence $x_1, \ldots, x_n$ of A. In the contrary case we say that S is *M-dependent*. It is easily seen that each subset of an M-independent set is, a fortiori, M-independent and, conversely, each subset of A containing an M-dependent subset of A is itself M-dependent.

In particular, an element a of A is called *self-dependent* if the one-element set $\{a\}$ is M-dependent in $\mathscr{A}$.

2. M-independence in lattices. By the *lattices operations* we mean (as usual) the operations which form the joins and the meets, respectively. (For the lattice-theoretical terminology, see [1]). In this section we suppose that no further fundamental operation is defined on the lattices in question. Accordingly, a lattice $\mathscr{L}$ defined on the set L will be denoted by $(L; \frown, \smile)$.

The lattice operations being idempotent, the single algebraic operation of one variable on a lattice $(L; \frown, \smile)$ is the identity operator of L. Consequently:

THEOREM 1. *In a lattice $(L; \frown, \smile)$ there is no self-dependent element.*

Further, it is easy to see that in a lattice $(L; \frown, \smile)$ there are exactly four algebraic operations of two variables: the lattice operations $\frown$, $\smile$, and the selector operations $s_1^{(2)}$, $s_2^{(2)}$. Using this fact, we prove

THEOREM 2. *If $\{a_1, a_2\}$ is a totally unordered* (1) *subset of the lattice $\mathscr{L} = (L; \frown, \smile)$, then it is M-independent in $\mathscr{L}$.*

Indeed, if a_1 and a_2 are incomparable, then $a_1 \neq a_2$ and $a_1 \frown a_2 < a_j < a_1 \smile a_2$ $(j = 1, 2)$ or, in other terms,

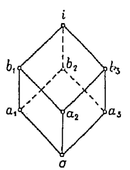
$$s_1^{(2)}(a_1, a_2) \neq s_2^{(2)}(a_1, a_2),$$

and

$$a_1 \frown a_2 < s_j^{(2)}(a_1, a_2) < a_1 \smile a_2 \qquad (j = 1, 2).$$

Hence, $\{a_1, a_2\}$ is M-independent in $\mathscr{L}$.

For a subset S of L with $\overline{\overline{S}} > 2$ the statement of Theorem 2 does not hold any more (2). For example, the subset $\{b_1, b_2, b_3\}$ of the lattice given by the diagram on the left is totally unordered and



$$b_1 \frown (b_2 \smile b_3) = s_1^{(3)}(b_1, b_2, b_3),$$

without that this equation be identically true. In fact,

$$a_1 \frown (a_2 \smile a_3) \neq s_1^{(3)}(a_1, a_2, a_3).$$

Thus the subset $\{b_1, b_2, b_3\}$ is M-dependent.

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(1) A subset S of a lattice is called *totally unordered* if the elements of S are pairwise incomparable.

(2) $\overline{\overline{S}}$ denotes the power of S.

The preceding example shows that the property of being totally unordered does not imply M-independence in general. We prove that the converse implication is always true. More generally:

THEOREM 3. *If S $(\overline{\overline{S}} \geqslant 2)$ is an M-independent subset of a lattice $(L; \frown, \smile)$, then for each subset $\{a_1, \ldots, a_n\}$ $(n \geqslant 2)$ of S*

$$(1) \qquad a_1 \smile \ldots \smile a_{k-1} \text{ non} \geqslant a_k \qquad (k = 2, \ldots, n),$$

and

$$(2) \qquad a_1 \frown \ldots \frown a_{k-1} \text{ non} \leqslant a_k \qquad (k = 2, \ldots, n).$$

COROLLARY 1. *Each M-independent subset of a lattice is totally unordered.*

COROLLARY 2. *No M-independent subset of a lattice $\mathscr{L}$ contains either the greatest or the least element of $\mathscr{L}$.*

COROLLARY 3. *Let $\mathscr{L}$ be a lattice with the least element. If S is an M-independent subset of atoms of $\mathscr{L}$, then $\overline{\overline{S}} \leqslant 2$.*

Proof. We prove Theorem 3 by indirect way. Suppose

$$a_1 \smile \ldots \smile a_{k-1} \geqslant a_k$$

for some k $(2 \leqslant k \leqslant n)$. Then we have

$$(a_1 \smile \ldots \smile a_{k-1}) \frown a_k = a_k = s_k^{(k)}(a_1, \ldots, a_k).$$

On the other hand, the equation

$$(x_1 \smile \ldots \smile x_{k-1}) \frown x_k = s_k^{(k)}(x_1, \ldots, x_k)$$

does not hold identically, for if we take $x_1 = \ldots = x_{k-1} < x_k$, then we get

$$(x_1 \smile \ldots \smile x_{k-1}) \frown x_k = x_1 \neq x_k = s_k^{(k)}(x_1, \ldots, x_k).$$

Consequently, $\{a_1, \ldots, a_n\}$ would be M-dependent, in contradiction to the fact that it is a subset of the M-independent set S.

By the dual arguments, the negation of (2) leads to a contradiction. Thus Theorem 3 is proved.

Since $a_j \geqslant a_k$, resp. $a_j \leqslant a_k$, with $j < k$, implies a fortiori

$$a_1 \smile \ldots \smile a_j \smile \ldots \smile a_{k-1} \geqslant a_k \quad \text{resp.} \quad a_1 \frown \ldots \frown a_j \frown \ldots \frown a_{k-1} \leqslant a_k,$$

Corollary 1 follows immediately from (1) and (2). Corollary 2 is a direct consequence of Corollary 1.

Finally, if $P = \{p_1, p_2, p_3, \ldots\}$ is a set of different atoms and o denotes the least element of $\mathscr{L}$, then $p_1 \frown p_2 = o < p_3$, that is, P does not satisfy (2). Thus, by Theorem 3, P is not M-independent.

The usual notion of independence concerning lattices with dimension function will be called here *L-independence* [3]. The lattice given by the diagram above shows that *L-independence does not imply M-independence and conversely*. In fact, the subset $\{a_1, a_2, a_3\}$ of this lattice is L-independent, but by the Corollary 3 to Theorem 3 it is not M-independent; conversely, the subset $\{b_1, b_2\}$ is, by Theorem 2, M-independent without being L-independent.

3. M-independence in lattices with unique complements. Let $\mathcal{L} = (L; \cap, \cup)$ be a lattice with greatest and least elements in which every element x has a unique complement x'. We consider the complementation as a third fundamental operation on $\mathcal{L}$ and therefore we write $\mathcal{L} = (L; \cap, \cup, ')$.

THEOREM 4. *The least and the greatest elements of a lattice $\mathcal{L} = (L; \cap, \cup, ')$ $(\overline{\overline{L}} \geqslant 2)$ are self-dependent but no further element of $\mathcal{L}$ is self-dependent.*

Proof. It is easy to see that the different algebraic operations of one variable on $\mathcal{L}$ are: the identity operation $s(x) = x$, the complementation operation $c(x) = x'$ and the constant operations

$$c_1(x) = x \cap x' = o, \quad c_2(x) = x \cup x' = i \quad (x \in L),$$

where o and i denote the least and the greatest element of $\mathcal{L}$, respectively. By the assumption $\overline{\overline{L}} \geqslant 2$, $s(x) \neq c(x)$, and $c_1(x) \neq c_2(x)$, for all x in L. Hence, Theorem 4 follows by the facts that $s(x) = c_1(x)$ or $c(x) = c_2(x)$ if and only if $x = o$ and $s(x) = c_2(x)$, or $c(x) = c_1(x)$ if and only if $x = i$.

THEOREM 5. *If a and b are elements of a lattice $\mathcal{L} = (L; \cap, \cup, ')$ such that a' and b are comparable, then the set $\{a, b\}$ is M-dependent in $\mathcal{L}$.*

By the lattice theoretical duality it is sufficient to consider the case $a' \leqslant b$. But then $a' \cup b = b = s_2^{(2)}(a', b)$ and $x' \cup y \neq s_2^{(2)}(x', y)$ in general. (Take, for example, $x = y \neq i$).

Remark. Theorem 5 shows that, in general, the statement of Theorem 2 does not hold if we consider the complementation as a third fundamental operation. In fact, if $\overline{\overline{L}} \geqslant 2$, then we can find an element b in L different from o and i; if we take now $a = b'$, then $a' = b$ too, and so a, b are incomparable elements, whereas $\{a, b\}$ is M-dependent by Theorem 5.

It may be asked whether the following converse of Theorem 5 holds: If a and b are elements of a lattice $\mathcal{L} = (L; \cap, \cup, ')$ such that neither a and b, nor a' and b, nor a and b', nor a' and b' are comparable, then $\{a, b\}$ is M-independent in $\mathcal{L}$ (**P 387**). We call the attention of the reader to the fact (see [1], p. 171) that a lattice $\mathcal{L} = (L; \cap, \cup, ')$ is either distributive or non-modular. In the case of distributivity, i. e. if $\mathcal{L}$ is a Boolean algebra, the considered converse is obviously true (e. g. in view of theorem 4 (i) of [3], p. 140).

4. M-dependence in semilattices. Let $\mathcal{S} = (S; \cap)$ be a semilattice and let $a \leqslant b$ $(a, b \in S)$ mean $a \cap b = a$. Using this partial ordering we give, a complete characterization for M-dependence in semilattices.

THEOREM 6 [4]. *Let $\mathcal{S} = (S; \cap)$ be a semilattice and T a subset of S. Then T is M-dependent in $\mathcal{S}$ if and only if there exist different elements $a_1, \ldots, a_r$ in T such that*

$$a_1 \cap \ldots \cap a_{r-1} \leqslant a_r. \tag{3}$$

COROLLARY. *Each M-independent subset of a semilattice is totally unordered.*

Proof. Since the fundamental operation of $\mathcal{S}$ is idempotent, the algebraic operations of n variables on $\mathcal{S}$ are the operations

$$f^{(n)}_{i_1, \ldots, i_p}(x_1, \ldots, x_n) = x_{i_1} \cap \ldots \cap x_{i_p},$$

where $(i_1, \ldots, i_p)$ is a fixed p-tuple of integers with $1 \leqslant i_1 < \ldots < i_p \leqslant n$. Moreover, if $\overline{\overline{S}} \geqslant 2$ and

$$(i_1, \ldots, i_p) \neq (j_1, \ldots, j_q) \quad (1 \leqslant i_1 < \ldots < i_p < n;\ 1 \leqslant j_1 < \ldots < j_q \leqslant n),$$

then

$$f^{(n)}_{i_1, \ldots, i_p} \neq f^{(n)}_{j_1, \ldots, j_q}. \tag{4}$$

In order to prove this assertion, consider an integer s $(1 \leqslant s \leqslant q)$ such that $j_s \neq i_1, \ldots, i_p$ and take, for example,

$$x_{i_1} = \ldots = x_{i_p} = a,$$
$$x_{i_1} = \ldots = x_{j_{s-1}} = x_{j_{s+1}} = \ldots = x_{j_q} = a, \quad x_{j_s} = b < a$$

(the existence of an element b with the property $b < a$ follows easily by the assumption $\overline{\overline{S}} \geqslant 2$). Then

$$f^{(n)}_{i_1, \ldots, i_p}(x_1, \ldots, x_n) = a \quad \text{and} \quad f^{(n)}_{j_1, \ldots, j_q}(x_1, \ldots, x_n) = b,$$

which proves (4).

Consequently, if T is an M-dependent subset in $\mathcal{S}$, then there exist different elements $b_1, \ldots, b_n$ in S such that

$$f^{(n)}_{i_1, \ldots, i_p}(b_1, \ldots, b_n) = f^{(n)}_{j_1, \ldots, j_q}(b_1, \ldots, b_n),$$

i. e.

$$b_{i_1} \cap \ldots \cap b_{i_p} = b_{j_1} \cap \ldots \cap b_{j_q} \quad \text{with} \quad (i_1, \ldots, i_p) \neq (j_1, \ldots, j_q).$$

(3) For the definition of this notion, see [1], p. 104.

(4) The proposition (iii) on p. 143 of [3] is a special case of Theorem 6, where $\mathcal{S}$ is the class of all subsets of a set.

Let s $(1 \leqslant s \leqslant q)$ be chosen so that $j_s \neq i_1, \ldots, i_p$. Then b_{j_s} differs from the elements $b_{i_1}, \ldots, b_{i_p}$ and

$$b_{i_1} \cap \ldots \cap b_{i_p} \leqslant b_{j_s}.$$

Hence, taking $p = r-1$, $a_k = b_{i_k}$ $(1 \leqslant k \leqslant r-1)$ and $a_r = b_{j_s}$, we find (3) satisfied.

Conversely, suppose that there exist elements $a_1, \ldots, a_r$ in $T(\subseteq S)$ such that (3) holds. Then

$$f^{(r)}_{1,\ldots,r}(a_1, \ldots, a_r) = f^{(r)}_{1,\ldots,r-1}(a_1, \ldots, a_r).$$

Since this equation does not hold identically, T is M-dependent.

Thus Theorem 6 and its Corollary are proved.

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CONCERNING THE INDEPENDENCE IN LATTICES

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The independence is meant here in the sense of [2] and [3]. The results presented here complete the paper [4] by Szász, in particular Theorem 1 is a strengthening of Theorem 3 of [4].

Nevertheless, the knowledge of Szász' paper is not necessary for the reader of this note.

The proof of Theorem 1 is a modification of Szasz' proof, made by J. Płonka.

1. Let us consider a lattice $(L; \cup, \cap)$.

THEOREM 1. *If I is a set of independent elements of L, then*

(i) $a_1 \cap \ldots \cap a_m$ non $\leqslant b_1 \cup \ldots \cup b_n$ *for each sequence* $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ $(m \geqslant 1, n \geqslant 1)$ *of different elements of* L [1].

Proof. Let us suppose

$$\bigcap_{j=1}^{m} a_j \leqslant \bigcup_{j=1}^{n} b_j$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n$ is a sequence of different elements of L. Hence

$$(*) \qquad \bigcap_{j=1}^{m} a_j \cup \bigcup_{j=1}^{n} b_j = \bigcup_{j=1}^{n} b_j.$$

Let us consider the following algebraic operations in L (= lattice polynomials):

$$f(x_1, \ldots, x_m, y_1, \ldots, y_n) = \bigcap_{j=1}^{m} x_j \cup \bigcup_{j=1}^{n} y_j,$$

and

$$g(x_1, \ldots, x_m, y_1, \ldots, y_n) = \bigcup_{j=1}^{n} y_j.$$

(1) The condition (i) for sets has been formulated by Tarski [5], p. 61. In this case (i) is equivalent to a condition treated in [3], p. 141, theorem (iii).