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SOME PROBLEMS IN THE ALGEBRA OF BOREL MEASURES

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Not much is known about the structure of the Banach algebra $M(G)$ composed of all complex-valued bounded regular Borel measures on an Abelian locally compact non-discrete group G (1). It may be recalled that addition in $M(G)$ and multiplication with a complex number are defined in the usual way, while the product of two elements is their convolution:

$$\mu \circ \nu(E) = \int_G \mu(E-t)\nu(dt)$$

for every Borel set $E \subset G$. Further, to define the norm of an element the smallest non-negative majorant $|\mu|$ of μ in $M(G)$ should be taken; then we put

$$\|\mu\| = |\mu|(G).$$

Let S be the space of all maximal ideals of $M(G)$ (Gelfand space) with the usual Gelfand-Stone topology. The form of these maximal ideals, in other words the form of an arbitrary homomorphism h of $M(G)$ onto the complex number field Z, was found by Šreĭder [6]. According to his result

$$h(\mu) = \int_G \overline{\chi_\mu(t)}\,\mu(dt) \quad \text{for every} \quad h \in S,$$

where the "generalized character" $\chi_\mu(t)$ is a function on $M(G)\times G$ subject to the following conditions:

(a) $\chi_\mu(t+u) = \chi_\mu(t)\chi_\mu(u)$ for almost every pair (t, u) in the sense of product measure $\mu\times\mu$ in $G\times G$,

(b) $\sup\limits_{\mu} \operatorname{es\,sup}\limits_{t\in G} |\chi_\mu(t)| = 1$,

(c) For every fixed μ the function $\chi_\mu(t)$ is μ-measurable and if $\nu \ll \mu$ (read: ν is absolutely continuous with respect to μ), then $\chi_\nu(t) = \chi_\mu(t)$ for ν-almost every t.

(1) See e. g. E. Hewitt, *A survey of abstract harmonic analysis*, in [3], especiall p. 132-149.

Our aim is to pose some questions regarding the classification of measures in $M(G)$ in terms of their behaviour as functions $\tilde{\mu}(h) = h(\mu)$ on S. Instead of $\tilde{\mu}(h)$ we shall also use the notation $(\mu)^{\sim}(h)$.

The most important measures are invariant measures (Haar measures); it is well-known that they differ from one another at most by a constant factor and they belong to $M(G)$ if and only if G is compact. A measure which is absolutely continuous with respect to Haar measure will be called briefly *absolutely continuous*. Such measures form a closed ideal I in $M(G)$. Plainly, if I is considered as a Banach algebra, it is isomorphic to the group algebra $L^1(G)$, and so the corresponding Gelfand space consists exactly of the Fourier transforms of L^1, or which means the same, of the Fourier-Stieltjes transforms of I. Thus it is identic with the dual group $\hat{G} = X$ of the group G and it is always (for non-discrete G) a proper subset of S. The Pontryagin topology in X is equivalent to the topology induced by S ([5], p. 50).

Definition 1. A measure μ will be called *normal*, if its spectrum $\{\tilde{\mu}(h): h \in S\}$ is contained in the set $\{\hat{\mu}(x): x \in X\} \cup (0)$, $\hat{\mu}$ or $(\mu)^{\wedge}$ denoting the Fourier-Stieltjes transform of μ.

Definition 2. If A is a Banach subalgebra of $M(G)$ and if, for every $h \in S$, $h(A) \neq 0$ implies the existence of an $x \in X$ such that $\tilde{\mu}(h) = \hat{\mu}(x)$ for all $\mu \in A$, then we call A *normal subalgebra*.

Let A be a Banach subalgebra of $M(G)$. According to whether A contains a unit element or not its Gelfand space S_A consists either of all maximal ideals of A or of all regular maximal ideals of A. In the first case S_A is compact, in the second it is locally compact non-compact ([4], § 11). An arbitrary homomorphism $M(G) \to Z$ can be restricted to A. By such restriction every element of S becomes an element of S_A. Obviously, in this way different elements of S can produce the same element of S_A; in particular, different elements of X can "melt together". We therefore introduce the space X_A which arises by identifying those elements x_1, x_2 of X for which $\hat{\mu}(x_1) = \hat{\mu}(x_2)$ whenever $\mu \in A$. In this way X_A becomes a subset of S_A. The topology in X_A derived from X by such identification is in general stronger than that induced by S_A. E. g., if G is the circle group and A consists exactly of measures vanishing outside the subgroup of elements of finite order, then $X_A = X$ (there is nothing to be identified) but the topology in X is discrete, whereas X_A is a dense subgroup of the compact group S_A (see e. g. [1]). On the other hand, if H is a *closed* subgroup of G and A consists exactly of measures vanishing outside H, then $X_A = \hat{H} = X/\mathfrak{A}_H$, $\mathfrak{A}_H$ being the annihilator of H, and so the (Pontryagin) topology in X_A, derived from X, is equivalent to that induced by S_A. In the sequel, whenever the space X_A is referred to as a subset of S_A, this inclusion will be considered in the topological sense as well.

Definition 3. If $S_A = X_A$, then we call A *strongly normal subalgebra*.

Definition 4. If for a measure $\mu \in M(G)$, for every $h \in S$ and for every $\varepsilon > 0$ there exists an $x \in X$ such that $|\tilde{\mu}(h) - \hat{\mu}(x)| < \varepsilon$, then we call μ an *analytic measure*: in other words, the measure μ is analytic if its spectrum is identical with the closure of $\hat{\mu}(X)$.

Definition 5. If X_A is dense in S_A, then A may be called *analytic subalgebra*.

Definition 6. If every homomorphism $h_1 \in S_A$ is extendable to a homomorphism h of the whole algebra $M(G)$ so that $h \in \bar{X}$, $\bar{X}$ denoting the closure of X in S, then A will be called *strongly analytic subalgebra*.

Note the following evident implications:

Every strongly normal subalgebra is normal and every strongly analytic subalgebra is analytic.

Every strongly normal subalgebra is strongly analytic.

If μ is a normal measure such that either $h(\mu) \neq 0$ for all $h \in S$ or 0 is in the closure of $\hat{\mu}(X)$, then μ is analytic.

If A is normal (analytic) and $\mu \in A$, then μ is normal (analytic).

Let A_μ be the smallest closed subalgebra containing the measure μ. If μ is normal, then A_μ is normal. If μ is analytic, then A_μ is analytic.

From the properties of the ideal I mentioned above we deduce at once that I is a strongly normal subalgebra; hence the absolutely continuous measures are normal. To have further examples of normal measures let us notice that the subalgebra consisting of measures vanishing outside a fixed finite subgroup of G is strongly normal. An example of an analytic subalgebra which is not normal may be $M_d = M_d(G)$ or the direct sum $M_d + I$, where M_d denotes the subalgebra composed of all totally discontinuous measures [2]. A curious example of a strongly analytic subalgebra can be found in [2] where the authors consider measures with supports contained in a very specifically constructed set Q, homeomorphic to the Cantor ternary set. It is proved that every linear functional L on the (linear) set $M(Q)$ of all measures under consideration can be extended to a homomorphism $h \in \bar{X}$, provided that L fulfils some simple conditions which are obviously necessary. It follows at once that the subalgebra generated by $M(Q)$ is strongly analytic.

Problem 1. Must a normal subalgebra be strongly normal (**P 394**)?

Problem 2. Must an analytic subalgebra be strongly analytic (**P 395**)?

Problem 3. Is $M_d(G)$ strongly analytic (**P 396**)?

We shall prove that the answer to the last question is positive in the case when G is the circle group $T = \{e^{it}: 0 \leqslant t < 2\pi\}$. Let h_1 be

(2) loc. cit. (1).

a homomorphism of $M_d(T)$ onto Z and let h be its extension defined as follows: if $\mu^{(d)}$ is the discrete and $\mu^{(c)}$ the continuous component of $\mu \in M(T)$, then put $h(\mu) = h_1(\mu^{(d)})$. So we have $h(\mu) = 0$ for every continuous measure. Since the set of all continuous measures is always an ideal in $M(G)$, h is actually a homomorphism. We must show that h is in $\bar{X}$. If $\mu_r = \mu_r^{(d)} + \mu_r^{(c)}$ $(r = 1, \ldots, k)$ are measures from $M(T)$ written as sums of their discrete and continuous components, then let $H = (t_1, t_2, \ldots)$ be the subgroup of T generated by the supports of $\mu_1^{(d)}, \ldots, \mu_k^{(d)}$. There is a character (in general not continuous) ψ_h of H such that

$$h(\mu_r) = \sum_{j=1}^{\infty} \mu_r^{(d)}(t_j)\psi_h(t_j) \qquad (r = 1, \ldots, k).$$

Thus, for an $\varepsilon > 0$ a continuous character e^{int} of T has to be found so that for $r = 1, \ldots, k$ the following inequalities hold:

$$\text{(i)} \qquad \Big| h(\mu_r) - \sum_{j=1}^{\infty} \mu_r^{(d)}(t_j) e^{int_j} \Big| < \varepsilon,$$

$$\text{(ii)} \qquad |a_n^{(r)}| < \varepsilon \qquad \Big(a_n^{(r)} = \int_0^{2\pi} e^{-int} \mu_r^{(c)}(dt)\Big).$$

If N is sufficiently large and δ sufficiently small, then (i) will be implied by

$$\text{(iii)} \qquad |e^{int_j} - \psi_h(t_j)| < \delta \qquad (j = 1, \ldots, N).$$

In virtue of Kronecker theorem (iii) is satisfied by some positive integers $n_1, n_2, \ldots$, the sequence of which has positive density. Now the Fourier-Stieltjes coefficients a_n of a continuous measure are for $n \neq 0$ equal to those of a continuous function of bounded variation. Hence, applying the well-known theorem of Wiener [8], we have

$$\lim_{n\to\infty} \frac{1}{n} \{|a_1^{(r)}| + \ldots + |a_n^{(r)}|\} = 0 \qquad (r = 1, \ldots, k).$$

Consequently

$$\lim_n \frac{1}{n} \Big\{ \sum_{r=1}^{k} |a_1^{(r)}| + \ldots + \sum_{r=1}^{k} |a_n^{(r)}| \Big\} = 0.$$

Obviously, among the n_j there is a number n_0 such that $\sum_{r=1}^{k} |a_{n_0}^{(r)}| < \varepsilon$. Hence we can satisfy (ii) and (iii) simultaneously taking $n = n_0$.

Analogously it can be proved that $M_d(G)$ is strongly analytic if G is the real axis.

Here are further questions concerning normality and analyticity:

PROBLEM 4. If a subalgebra consists of normal measures only, must it be normal (**P 397**)?

PROBLEM 5. If a subalgebra consists of analytic measures only must it be analytic (**P 398**)?

PROBLEM 6. Is the set of all normal measures a subalgebra (**P 399**)?

PROBLEM 7. Is the set of all analytic measures a subalgebra (**P 400**)?

PROBLEM 8. If μ is normal (analytic) and $\nu \ll \mu$, must then ν be normal (analytic) (**P 401**)?

The unique possible involution in $M(G)$ is given by $\mu(E) \to \overline{\mu(E^{-1})} = \mu^*(E)$.

Definition 7. A measure will be called *symmetric* if $\overline{\tilde{\mu}(h)} = (\mu^*)^{\sim}(h)$ for all $h \in S$.

It is easily seen that μ and μ^* are either both symmetric or both assymetric.

Definition 8. The subalgebra A, closed with respect to involution, is called *symmetric* if it consists of symmetric measures only. It is called *strongly symmetric* if $\overline{\tilde{\mu}(h)} = (\mu^*)^{\sim}(h)$ for all $h \in S_A$ and $\mu \in A$.

Since $\bar{\hat{\mu}} = (\mu^*)^{\hat{}}$, a subalgebra closed with respect to involution must be symmetric if it is normal and strongly symmetric if it is analytic. It is a known fact that the whole algebra $M(G)$ is assymetric and so not analytic ([3]). Actually, there are examples of measures in $M(G)$ such that $|\hat{\mu}(x)| > \delta > 0$ but $1/\hat{\mu}(x)$ is no Fourier-Stieltjes transform of any measure. According to the well known theorem of Mazur and Gelfand this phenomenon cannot occur if μ is an analytic measure. Moreover: if μ is analytic and $F(z)$ is a holomorphic function in a region containing the spectrum of μ, then there exists a measure μ_F such that $\tilde{\mu}_F(h) = F(\tilde{\mu}(h))$ $(h \in S)$, and therefore $\hat{\mu}_F = F(\hat{\mu})$. This fact justifies perhaps the term "analytic measure".

PROBLEM 9. Must a symmetric subalgebra be strongly symmetric (**P 402**)?

PROBLEM 10. Is every symmetric or every strongly symmetric subalgebra analytic (**P 403**)?

PROBLEM 11. Is every symmetric measure analytic or conversely (**P 404**)?

Obviously, if μ is analytic and if $\hat{\mu}(x) \to 0$ (i. e. if $|\hat{\mu}(x)| < \varepsilon$ outside a compact subset of X), then μ is normal.

([3]) loc. cit. ([1]).

PROBLEM 12. Is every measure μ with $\hat{\mu}(x) \to 0$ a normal measure? A symmetric measure (**P 405**)?

PROBLEM 13. Is for a normal continuous measure always $\hat{\mu}(x) \to 0$ (**P 406**)?

It is easily seen that if A is a strongly normal subalgebra without unit, then $\hat{\mu}(x) \to 0$ for all $\mu \in A$. In fact, let us notice first that the smallest subalgebra A_a containing A and all absolutely continuous measures (i. e. the ideal I) will be still strongly normal: $S_{A_a} = X_{A_a}$. Further, we have $X_{A_a} = X$ and the topology induced in X_{A_a} by S_{A_a} is the same as the "original" topology of the dual $\hat{G} = X$. It is sufficient to observe that I alone provides such topology. Since A_a has no unit, S_{A_a} is locally compact non compact and $|\hat{\mu}(x)| < \varepsilon$ outside a compact set in $S_{A_a} = X$.

PROBLEM 14. If $\hat{\mu}(x) \to 0$ for every $\mu \in A$, must then A be a normal or even a strongly normal subalgebra (**P 407**)?

Obviously, a positive answer even to the weaker one of these questions would imply a positive solution of Problem 12. If the solution of Problem 14 is "strongly positive", then the ideal consisting of all measures for which $\hat{\mu}(x) \to 0$ would be the largest strongly normal subalgebra without unit.

Definition 9. A measure μ may be called *Cauchy measure* if for any μ-measurable function φ, not equal to 0 μ-almost everywhere, the relation

$$\varphi(t+u) = \varphi(t)\varphi(u)$$

for μ-almost every element (t, u) of $G \times G$ implies the existence of a continuous character x of the group G such that $x(t) = \varphi(t)$ μ-almost everywhere.

It is known that Haar measure μ_0 is a Cauchy measure (see e. g. [1]) and so is consequently every measure ν such that $\mu_0 \ll \nu$ and $\nu \ll \mu_0$. It follows from the property (a) of a generalized character χ_μ that if μ is a Cauchy measure, then $\chi_\mu(t)$ is μ-almost everywhere equal to a character and $\int \overline{\chi_\mu(t)} \mu(dt)$ is a value of $\hat{\mu}(x)$. Hence every Cauchy measure is normal. Moreover, if μ is a Cauchy measure, then by (c) every measure $\nu \ll \mu$ is normal. The measure σ in T induced by the Lebesgue step function is not a Cauchy measure, since there exists a generalized character χ_μ such that $0 < |\chi_\sigma(t)| = \text{constans} < 1$ σ-almost everywhere. This example was produced by Šreĭder in [7] and the present author apologizes for having overlooked it and mentioned in [1] the Cauchy property of measure σ as an open question. The matters under discussion can be an object of further problems, as e. g.:

PROBLEM 15. Is σ an analytic or even a normal measure (**P 408**)?

PROBLEM 16. Is every absolutely continuous measure a Cauchy measure (**P 409**)?

It may be noticed that there are normal measures without Cauchy property, for example the point measure in $T = \{e^{it}: 0 \leqslant t < 2\pi\}$ placed at $t = \pi$.

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