only if the space $\mathfrak{M}$ may be written in the form (1.2) *of union of two disjoint closed subsets. The relations between these two decompositions are given by* (1.3), (1.5) *and* (1.6).

Remark. The proof given in [4] cannot be indirectly used here, because we do not know whether the theory of analytic functions of several variables known for the Banach algebras is true for the p-normed algebras. So we pose the following problem:

P 393. Let A be a p-normed algebra, $\mathfrak{M}$ its multiplicative linear functional space. Let $x_1, \ldots, x_n \in A$. The joint spectrum of the n-tuple $(x_1, \ldots, x_n)$ is defined as

$$\sigma(x_1, \ldots, x_n) = \{(f(x_1), \ldots, f(x_n)) \in C^n : f \in \mathfrak{M}\}.$$

Let $\Phi(z_1, \ldots, z_n)$ be an analytic function of n complex variables defined on an open subset $U \subset C^n$ containing the spectrum $\sigma(x_1, \ldots, x_n)$. Does there exist in A an element y such that

$$f(y) = \Phi(f(x_1), \ldots, f(x_n)) \quad \text{for every} \quad f \in \mathfrak{M}?$$

A similar problem may be posed also for the locally analytic operations in a p-normed algebra (for the definition cf. [2], § 13).

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*ON A NEW APPROACH TO CONTINUOUS METHODS OF SUMMATION**

BY

L. WŁODARSKI (ŁÓDŹ)

Introduction. In my preceding paper [4] I gave a definition of the continuous methods of limitation as follows:

Definition 1. A functional method of limitation A described by the sequence $\{a_\nu(t)\}$ of functions $a_\nu(t)$ defined in the interval $t_0 \leqslant t < T$ $(T \leqslant +\infty)$ is called *continuous method* if

(i) all functions $a_\nu(t)$ are continuous in this interval $t_0 \leqslant t < T$,

(ii) there exists an increasing sequence $t_0, t_1, t_2, \ldots, t_m, \ldots$ tending to T such that for every sequence $x = \{\xi_\nu\}$ the convergence of the series

$$A(t, x) = \sum_{\nu=0}^{\infty} a_\nu(t)\, \xi_\nu$$

for $t = t_m$ and $t = t_{m+1}$ implies uniform convergence of the series $A(t, x)$ in the interval $t_m \leqslant t \leqslant t_{m+1}$.

Definition 2. The sequence $x = \{\xi_\nu\}$ is called *limitable by the continuous method A to the number ξ*, if

1° the series $A(t, x)$ is convergent for $t_0 \leqslant t < T$,

2° the limit $\lim\limits_{t \to T-} A(t, x) = \xi$ exists.

Definition 3. The set A^* of sequences $x = \{\xi_\nu\}$ limitable by the method A is called the *field of the method A*.

Now we shall give a new definition of a continuous method of limitation:

Definition 4. A functional method of limitation $A = \{a_\nu(t)\}$ $(t_0 \leqslant t < T)$ will be called *continuous method* (*in a new sense*) if this method satisfies the condition (i).

* written during my stay at Tulane University.

We have given up condition (ii) for continuous method of limitation, but we introduce instead of this an additional condition for the limitability of sequences; this stronger limitability shall be called *c-limitability*.

Definition 5. We shall call a sequence $x = \{\xi_v\}$ *c-limitable* to the value ξ by a continuous method A if, besides the conditions 1° and 2°, the following condition is satisfied:

3° the series $A(t, x)$ is almost uniformly convergent in the interval $t_0 \leqslant t < T$, i. e. uniformly convergent in every closed interval $t_0 \leqslant t \leqslant \bar{t}$, where $\bar{t} < T$.

Definition 6. The *c-field* A^* of the method A is the set of sequences c-limitable by this method.

In this way, for the sequences c-limitable by continuous method in the new sense we can obtain all the theorems proved for sequences limitable by continuous methods in the old sense (see [4], [5], [6]). Moreover thanks to this *such important methods as Riemann's are included in the general theory of continuous methods*. Also the c-field (def. 6) of the continuous method in the old sense (def. 1) is obviously the same as the field (def. 3) of this method, because the almost uniform convergence of the series $A(t, x)$ is a consequence of condition (ii).

1. We shall talk of continuous methods in the new sense only (definition 4).

THEOREM I. *Every sequence* $x = \{\xi_v\}$ *convergent to zero is c-limitable* (def. 5) *by the continuous method* A *to the number* $\sum a_v \xi_v$ *if and only if*

(a) $$\lim_{t \to T-} a_v(t) = a_v \qquad (v = 0, 1, 2, \ldots),$$

(b) $$|A| = \sup_{t_0 \leqslant t < T} \sum_{v=0}^{\infty} |a_v(t)| < +\infty.$$

Proof. Necessity. The condition (a) is obviously satisfied and (b) follows from the definition of c-limitability and from Theorem I in [4] (p. 165).

Sufficiency. The condition (a) and (b) imply also that $\sum_{v=0}^{\infty} |a_v| \leqslant |A|$. Now we shall prove for every sequence $\{\xi_v\}$ convergent to zero that the series $\sum_{v=0}^{\infty} a_v(t)\xi_v$ is uniformly convergent in the interval $t_0 \leqslant t < T$, and that $\lim_{t \to T-} \sum_{v=0}^{\infty} a_v(t)\xi_v = \sum_{v=0}^{\infty} a_v \xi_v$. This follows from our assumptions and the inequalities

$$\Big|\sum_{v=N+1}^{\infty} a_v(t)\xi_v\Big| \leqslant |A| \sup_{v>N} |\xi_v|$$

and

$$\Big|\sum_{v=0}^{\infty} a_v(t)\xi_v - \sum_{v=0}^{\infty} a_v \xi_v\Big| \leqslant \sup_v |\xi_v| \Big[\sum_{v=0}^{N} |a_v(t) - a_v|\Big] + \sup_{v>N} |\xi_v| \Big[\sum_{v=N+1}^{\infty} |a_v(t)| + \sum_{v=N+1}^{\infty} |a_v|\Big].$$

THEOREM II. *Every convergent sequence* $x = \{\xi_v\}$ *is c-limitable by the continuous method* A *to the number* $\sum_{v=0}^{\infty} a_v \xi_v + \big(a - \sum_{v=0}^{\infty} a_v\big) \lim_{v \to \infty} \xi_v$ *if and only if* (a) *and* (b) *hold and*

(c) *the series* $\sum_{v=0}^{\infty} a_v(t)$ *is almost uniformly convergent in* $\langle t_0, T)$,

(d) *the limit* $\lim_{t \to T} \sum_{v=0}^{\infty} a_v(t) = a$ *exists.*

Proof. Necessity is obvious.

Sufficiency. Let $\bar{x}$ denote $\{\xi_v - \xi\}$, where $\xi = \lim_{v \to \infty} \xi_v$; then

$$A(t, x) = \sum_{v=0}^{\infty} a_v(t)\xi_v = A(t, \bar{x}) + \xi \sum_{v=0}^{\infty} a_v(t).$$

Our assumptions and theorem I imply, that the series $A(t, x)$ is almost uniformly convergent in $\langle t_0, T)$, and it is obvious that

$$\lim_{v \to T-} A(t, x) = \sum_{0}^{\infty} a_v(\xi_v - \xi) + \xi a = \sum_{=0}^{\infty} a_v \xi_v + \xi\Big(a - \sum_{v=0}^{\infty} a_v\Big).$$

Definition 7. A continuous method of limitation A is called *c-permanent*, if every sequence $x = \{\xi_v\}$ convergent to ξ is also c-limitable to ξ.

THEOREM III. *The continuous method* A *is c-permanent if and only if conditions* (a), (b), (c), (d) *are satisfied, and* $a_v = 0$ $(v = 0, 1, 2, \ldots)$, $a = 1$.

Proof. Evident.

2. Now we shall consider *continuous methods for series*.

Definition 8. Let B denote a method described by a sequence $b_v(t)$, where $b_v(t)$ are continuous functions in the interval $t_0 \leqslant t < T$.

Definition 9. We shall call the series Γ: $\sum_{v=0}^{\infty} \gamma_v$ *c-summable* by the continuous method B (def. 8) to the number ξ if

1° the series $B(t, \Gamma) = \sum_{v=0}^{\infty} b_v(t)\gamma_v$ is almost uniformly convergent in $\langle t_0, T)$, i. e. uniformly convergent in every interval $\langle t_0, \bar{t}\rangle$ where $\bar{t} < T$,

2° the limit $\lim_{t\to T-} B(t, \Gamma) = \xi$ exists.

THEOREM IV. *Every convergent series* $\sum_{v=0}^{\infty}\gamma_v$ *is c-summable by the continuous method B to the number* $\sum_{v=0}^{\infty}\beta_v\gamma_v$ *if and only if*

(A) *the limits* $\lim_{t\to T-} b_v(t) = \beta_v$ $(v = 0, 1, 2, \ldots)$ *exist,*

(B) $|B| = \sup_{t_0\leqslant t<T} \sum_{v=0}^{\infty} |b_v(t) - b_{v+1}(t)| < \infty$.

Proof. The necessity of the condition (A) is obvious. We shall prove (B). From our assumption it follows that the functional $B(t, \Gamma) = \sum_{v=0}^{\infty} b_v(t)\gamma_v$ is, in particular, well defined for every $\Gamma \epsilon l$ (i. e. for every $\Gamma = \sum\gamma_v$ such that $\sum|\gamma_v| < \infty$) when t is fixed. It is well-known, that in the space l the norm of the functional is $\|B(t, \Gamma)\|_l = \sup_v |b_v(t)|$ ([1], p. 67). On the other hand, since $\lim_{t\to T-} B(t, \Gamma)$ exists, it follows from our assumptions that for a fixed Γ the function $B(t, \Gamma)$ is continuous in $t_0 \leqslant t < T$ and therefore it is also bounded, i. e.

$$(*) \qquad \sup_{t_0\leqslant t<T} |B(t, \Gamma)| < \infty.$$

But inequality (*) given above means that the family of the functionals $B(t, \Gamma)$, for variable t, is bounded for every $\Gamma \epsilon l$, and from the Banach-Steinhaus theorem ([1], théorème 5, p. 80) it follows that their norms are also all together bounded, i. e. $B^* = \sup_{v,t} |b_v(t)| < \infty$.

Using the Abel transformation

$$\sum_{v=m}^{n} b_v(t)\gamma_v = \sum_{v=m}^{n-1} [b_v(t) - b_{v+1}(t)]\,\xi_v - b_m(t)\,\xi_{m-1} + b_n(t)\,\xi_n,$$

where $\xi_v = \gamma_0 + \gamma_1 + \ldots + \gamma_v$, we see that the series $\sum_{v=0}^{\infty}[b_v(t) - b_{v+1}(t)]\,\xi_v$ is almost uniformly convergent in $\langle t_0, T)$ for every sequence $\{\xi_v\}$ convergent to zero, and for these sequences

$$\lim_{t\to T-} \sum_{v=0}^{\infty} [b_v(t) - b_{v+1}(t)]\,\xi_v$$

exists, because

$$\sum_{v=0}^{\infty} b_v(t)\gamma_v = \sum_{v=0}^{\infty} [b_v(t) - b_{v+1}(t)]\,\xi_v.$$

Hence (B) follows from theorem I.

Sufficiency. Since

$$b_n(t) = b_0(t) + \sum_{v=0}^{n-1} [b_{v+1}(t) - b_v(t)]$$

it follows from the condition (B) that

$$\sup_{n, t_0\leqslant t<T} |b_n(t)| \leqslant B^{**}, \quad \text{where} \quad B^{**} = |B| + \sup_{t_0\leqslant t<T} |b_0(t)| < \infty.$$

We can write the Abel transformation in the form

$$\sum_{v=m}^{n} b_v(t)\gamma_v = \sum_{v=m}^{n-1} [b_v(t) - b_{v+1}(t)][\xi_v - \xi] - b_m(t)[\xi_{m-1} - \xi] + b_n(t)[\xi_n - \xi],$$

where $\xi_v = \gamma_0 + \gamma_1 + \ldots + \gamma_v$ and $\xi = \lim_{v\to\infty} \xi_v$. Hence we see that the series $B(t, \Gamma) = \sum_{v=0}^{\infty} b_v(t)\gamma_v$ is uniformly convergent in $\langle t_0, T)$ for every convergent series $\Gamma = \sum\gamma_v$, and

$$B(t, \Gamma) = \sum_{v=0}^{\infty} b_v(t)\gamma_v = \sum_{v=0}^{\infty} [b_v(t) - b_{v+1}(t)][\xi_v - \xi] + \xi b_0(t).$$

As a consequence of this and of theorem I we have

$$\lim_{t\to T-} B(t, \Gamma) = \sum_{v=0}^{\infty} (\beta_v - \beta_{v+1})(\xi_v - \xi) + \xi\beta_0,$$

and hence, after easy calculations, we obtain finally

$$\lim_{t\to T-} B(t, \Gamma) = \sum_{v=0}^{\infty} \beta_v\gamma_v.$$

COROLLARY. *If every convergent series* Γ *is c-summable by the continuous method B, then for these series* Γ *the series* $B(t, \Gamma)$ *is uniformly convergent.*

Definition 10. A continuous method B of summation is called *c-permanent* if every series which sums to γ is c-summable to the same number γ.

THEOREM V. *A continuous method of summation B is c-permanent if conditions* (A) *and* (B) *from theorem* IV *hold for* $\beta_v = 1$ $(v = 0, 1, 2, \ldots)$.

Proof evident.

3. Definition 11. The set of the series $\Gamma = \sum\gamma_v$ which satisfy the condition 1° from definition 9 is called the *c-pseudofield* B_c^{**} of the continuous method B of summation.

THEOREM VI. *The c-pseudofield B_c^{**} of the continuous method B of summation (def. 8) is the B_0-space (or Mazur-Orlicz space, see [2]) with the pseudonorms*:

$$|\Gamma|_n = |\gamma_n| \quad (n = 0, 1, 2, \ldots),$$

$$|\Gamma|_m^B = \sup_{p, t_{m-1} \leqslant t \leqslant t_m} \Big|\sum_{v=0}^{p} b_v(t)\gamma_v\Big| \quad (m = 1, 2, 3, \ldots),$$

where $\{t_m\}$ is an increasing sequence which tends to T.

Proof. We have to prove that if a sequence $\{\Gamma^k\}$, where $\Gamma^k = \sum\gamma_v^k \in B_c^{**}$, satisfies Cauchy's condition, then there exists an element $\Gamma^0 \in B_c^{**}$ such that $\Gamma^k \to \Gamma^0$ in B_c^{**}. It is clear that provided our assumptions are satisfied the limits $\lim_{k\to\infty}\gamma_v^k = \gamma_v^0$ exist. We shall prove that $\Gamma^0 = \sum\gamma_v^0$ is the required element. We prove first that $\Gamma^k \to \Gamma^0$ in B_c^{**}, i. e. $\lim|\Gamma^k - \Gamma^0|_n = 0$ $(n = 0, 1, 2, \ldots)$ and $\lim|\Gamma^k - \Gamma^0|_m^B = 0$ $(m = 1, 2, 3, \ldots)$, using the inequalities $|\Gamma^k - \Gamma^p|_n < \varepsilon$ and $|\Gamma^k - \Gamma^p|_m^B < \varepsilon$ for large k and p. Then we have to prove that $\Gamma^0 \in B_c^{**}$. The almost uniform convergence of the series $B(t, \Gamma^0) = \sum_{v=0}^{\infty} b_v(t)\gamma_v^0$ in the interval $t_0 \leqslant t < T$ follows from the inequality

$$\sup_t \Big|\sum_{v=r+1}^{q} b_v(t)\gamma_v^0\Big| \leqslant \sup_t \Big|\sum_{v=0}^{r} b_v(t)(\gamma_v^0 - \gamma_v^k)\Big| + \\ + \sup_t \Big|\sum_{v=0}^{q} b_v(t)(\gamma_v^0 - \gamma_v^k)\Big| + \sup_t \Big|\sum_{v=r+1}^{q} b_v(t)\gamma_v^k\Big|,$$

by the relation $\Gamma^k \to \Gamma^0$ in B_c^{**}, proved above, and by the fact that $\Gamma^k \in B_c^{**}$.

Definition 12. The *c-field* B_c^* of the continuous method B of summation is the set of all series which are c-summable (def. 9) by the method B.

THEOREM VII. *The c-field B_c^* of the continuous method B of summation is the B_0-space with the pseudonorm*

$$|\Gamma|^B = \sup_{t_0 \leqslant t < T} |B(t, \Gamma)|$$

and the pseudonorms

$$|\Gamma|_n \quad (n = 0, 1, 2, \ldots), \qquad |\Gamma|_m^B \quad (m = 1, 2, 3, \ldots)$$

given in theorem VI.

Proof of this theorem is quite similar to the proof of the analogous theorem in [4] (th. IV, p. 173), based on the previous theorem VI.

Definition 13. Let X_0 denote the space of sequences $x = \{\xi_v(t)\}$ of functions $\xi_v(t)$, defined and continuous in the closed interval $\langle\alpha, \beta\rangle$ such that $\xi_v(t)$ tends uniformly to zero in this interval.

LEMMA 1. *The set X_0 (def. 13) is a Banach-space with the norm*

$$\|x\| = \sup_{v,t} |\xi_v(t)|.$$

Proof is obvious.

Definition 14. We call a functional *linear* if it is additive, homogeneous and continuous.

LEMMA 2. *The general form of a linear functional in the space X_0 (def. 13) with the norm $\|x\| = \sup_{v,t}|\xi_v(t)|$ is*

$$f(x) = \sum_{v=1}^{\infty} \xi_v(t)\,d\eta_v(t),$$

where

(i) *$\eta_v(t)$ is a function with bounded variation in $\langle\alpha, \beta\rangle$ for $v = 1, 2, \ldots$,*

(ii) *$\sum_{v=1}^{\infty} V_\alpha^\beta \eta_v < \infty$, where $V_\alpha^\beta \eta_v$ means the total variation of the function $\eta_v(t)$ on $\langle\alpha, \beta\rangle$,*

(iii) *$\eta_v(\alpha) = 0$ $(v = 1, 2, 3, \ldots)$.*

Proof. Let X_v denote the subspace of X_0 whose elements are

$$x_v = \{0, 0, \ldots, 0, \xi_v(t), 0, 0, \ldots\}$$

(i. e. X_v is the set of vectors whose v-th coordinate is the only one different from zero) with the norm $\|X_v\| = \sup_t|\xi_v(t)|$. Then it is easy to see, that the general form of a linear functional in X_v is $f(x_v) = \int_\alpha^\beta \xi_v(t)\,d\eta_v(t)$, where $\eta_v(t)$ is a function with bounded variation on $\langle\alpha, \beta\rangle$. This holds because the space X_v is equivalent to the space $C\langle\alpha, \beta\rangle$ of all continuous functions on closed interval $\langle\alpha, \beta\rangle$, and the general form of linear functional in $C\langle\alpha, \beta\rangle$ is well-known (see [1], p. 61).

On the other hand, in our space X_0 we have $\lim_{n\to\infty}\|x - \sum_{v=1}^{n} x_v\| = 0$; hence

$$f(x) = f\Big(\lim_{n\to\infty}\sum_{v=1}^{n} x_v\Big) = \lim_{n\to\infty} f\Big(\sum_{v=1}^{n} x_v\Big) = \lim_{n\to\infty}\sum_{v=1}^{n} f(x_v)$$

$$= \sum_{v=1}^{\infty} f(x_v) = \sum_{v=1}^{\infty}\int_\alpha^\beta \xi_v(t)\,d\eta_v(t),$$

where $\eta_v(t)$ are functions with bounded variation on $\langle\alpha, \beta\rangle$.

We see that

$$|f(x)| \leqslant \sup_{v,t}|\xi_v(t)| \cdot \sum_{v=1}^{\infty} V_\alpha^\beta \eta_v = \|x\| \cdot \sum_{v=1}^{\infty} V_\alpha^\beta \eta_v;$$

hence

$$\|f\| \leqslant \sum_{v=1}^{\infty} \mathrm{V}_a^\beta \eta_v.$$

On the other hand, it is easy to observe that we can choose $x_v = \{0, 0, \ldots, 0, \xi_v(t), 0, 0, \ldots\}$ such that $\|x_v\| \leqslant 1$, $|f(x_v)| \geqslant \mathrm{V}_a^\beta \eta_v - \varepsilon_v$, where ε_v is arbitrarily small (see e.g. [1], p. 59 and 60). Let y_n denote $\sum_{v=1}^{n} x_v$, where x_v are chosen as above; then we have

$$\|y_n\| = \sup_{1\leqslant v\leqslant n, t} |\xi_v(t)| \leqslant 1,$$

and hence

$$\|f\| \geqslant |f(y_n)| \geqslant \sum_{v=1}^{n} \mathrm{V}_a^\beta \eta_v - \sum_{v=1}^{n} \varepsilon_v.$$

Considering both inequalities obtained above we have, finally,

$$\|f\| = \sum_{v=1}^{\infty} \mathrm{V}_a^\beta \eta_v < \infty.$$

Of course, without losing generality, we can assume that $\eta_v(a) = 0$ for $v = 1, 2, 3, \ldots$

LEMMA 3. *The space X of uniformly convergent sequences $x = \{\xi_v(t)\}$ of continuous functions $\xi_v(t)$ on closed interval $\langle a, \beta\rangle$ is a Banach space with the norm* $\|x\| = \sup_{v,t} |\xi_v(t)|$.

Proof. If is easy to see that for every $x \epsilon X$ the value of $\|x\|$ is finite. The proof of the completeness of the space X is similar to the proofs of similar theorems. If $x^p = \{\xi_v^p(t)\} \epsilon X$ and $\{x^p\}$ is a Cauchy-sequence in X, then the sequence $\{\xi_v^p(t)\}$, when p tends to infinity (with v fixed), is uniformly convergent in t to a certain function $\xi_v(t)$. Let us write $x = \{\xi_v(t)\}$. It is easy to prove that $x^p \to x$ in X and that $x \epsilon X$.

LEMMA 4. *The general form of the linear* (*def.* 14) *functional* $f(x)$, $x = \{\xi_v(t)\} \epsilon X$ (*where X is the same as in lemma* 3), *is*

$$f(x) = \int_a^\beta \xi(t)\, d\eta(t) + \sum_{v=1}^{\infty} \int_a^\beta \xi_v(t)\, d\eta_v(t),$$

where $\eta(t)$, and $\eta_v(t)$ satisfy the condition (i), (ii), *and* (iii) *from lemma* 2, *and* $\xi(t) = \lim_{v\to\infty} \xi_v(t)$.

Proof. Let us consider $f(x_0)$, where $x_0 = \{\xi_v(t) - \xi(t)\} \epsilon X_0$ (def. 13). Since the norms $\|x\|$ in X_0 and in X are the same, it follows from lemma 2 that

$$f(x_0) = \sum_{v=1}^{\infty} \int_a^\beta [\xi_v(t) - \xi(t)]\, d\eta_v(t), \quad \text{where} \quad \sum_{v=1}^{\infty} \mathrm{V}_a^\beta \eta_v < \infty;$$

hence

$$f(x_0) = \sum_{v=1}^{\infty} \int_a^\beta \xi_v(t)\, d\eta_v(t) - \int_a^\beta \xi(t)\, d \sum_{v=1}^{\infty} \eta_v(t).$$

In virtue of conditions (ii) and (iii), $|\eta_v(t)| \leqslant \mathrm{V}_a^\beta \eta_v$ for $a \leqslant t \leqslant \beta$, and hence the series $\sum_{v=1}^{\infty} \eta_v(t)$ is uniformly convergent in this interval. The function $\sum_{v=1}^{\infty} \eta_v(t)$ has bounded variation because $\mathrm{V}_a^\beta \sum_{v=1}^{\infty} \eta_v \leqslant \sum_{v=1}^{\infty} \mathrm{V}_a^\beta \eta_v < \infty$.

On the other hand, we have $x = x_0 + \bar{x}$, where $\bar{x} = \{\xi(t), \xi(t), \xi(t), \ldots\}$. The subspace $\bar{X} \subset X$ whose elements $\bar{x}$ are as above is equivalent to the C-space of all continuous functions on the same interval. Hence for every f there exists an $\bar{\eta}(t)$ such that $f(\bar{x}) = \int_a^\beta \xi(t)\, d\bar{\eta}(t)$. Finally we have

$$f(x) = f(x_0) + f(\bar{x}) = \int_a^\beta \xi(t)\, d\eta(t) + \sum_{v=1}^{\infty} \int_a^\beta \xi_v(t)\, d\eta_v(t),$$

where $\eta_v(t)$ satisfy the conditions (i), (ii), (iii), and $\eta(t) = \bar{\eta}(t) - \sum_{v=1}^{\infty} \eta_v(t)$ is of bounded variation.

LEMMA 5. *The general form of the linear functional in B_c^{**} with respect to the pseudonorm*

$$\|\Gamma\|_m^B = \sup_{n; t_{m-1} \leqslant t \leqslant t_m} \Big| \sum_{v=0}^{n} b_v(t) \gamma_v \Big|$$

is as in lemma 4, *where* $a = t_{m-1}$, $\beta = t_m$, $\xi_v(t) = \sum_{r=0}^{v} b_r(t) \gamma_r$ *and* $\xi(t) = \sum_{r=0}^{\infty} b_r(t) \gamma_r$.

This follows from the fact that if we consider the quotient-space $B_c^{**}/\tilde{B}$, where $\tilde{B}$ is the subspace of B_c^{**} for which $\|\Gamma\|_m^B = 0$, this space with norm $\|\cdot\|_m^B$ is equivalent to the subspace of X considered in lemma 4 (here we use also the Hahn-Banach theorem about the extension of a linear functional [1], p. 27-29).

THEOREM VIII. *The general form of the linear* (*def.* 14) *functional in the c-pseudofield B_c^{**} of the continuous method B is*

$$f(\Gamma) = \sum_{k=0}^{\infty} c_k \gamma_k,$$

where c_k are arbitrary for $k \leqslant p$ (p a finite number) and

$$c_k = \int_a^\beta b_k(t)\, d\Big[\eta(t) + \sum_{v=k}^{\infty} \eta_v(t)\Big]$$

for $k > p$, *where* $t_0 \leqslant \alpha < \beta < T$, *and functions* $\eta(t)$ *and* $\eta_v(t)$ *satisfy the conditions* (i), (ii), *and* (iii) *from lemma* 2.

Proof. It is known (see [3], th. 2.21, p. 139), that in B_0-space every linear functional is of the m^{th} order (i. e. is continuous with respect to the first m pseudonorms), so that (see [3], th. 2.23, p. 139) it is the sum $\sum_{v=1}^{m} f_v(\Gamma)$ of m functionals such that $f_v(\Gamma)$ is continuous with respect to the pseudonorm $\|\Gamma\|_v$. From lemma 5 and from the form of pseudonorms $\|\Gamma\|_m^B$ given in theorem VI it follows that

$$f(\Gamma) = \sum_{v=0}^{p} \bar{c}_v \gamma_v + \int_{\alpha}^{\beta} \sum_{v=0}^{\infty} b_v(t)\gamma_v \, d\eta(t) + \sum_{n=0}^{\infty} \int_{\alpha}^{\beta} \Big(\sum_{v=0}^{n} b_v(t)\gamma_v \Big) d\eta_n(t),$$

where $t_0 \leqslant \alpha < \beta < T$, $\eta(t)$ and $\eta_v(t)$ satisfy the conditions (i), (ii), and (iii), given in lemma 2, and $\bar{c}_v$ are arbitrary fixed numbers.

It is easy to see that

$$\int_{\alpha}^{\beta} \sum_{v=0}^{\infty} b_v(t)\gamma_v \, d\eta(t) = \sum_{v=0}^{\infty} \gamma_v \int_{\alpha}^{\beta} b_v(t)\, d\eta(t)$$

because the difference

$$\sigma_n = \sum_{v=0}^{n} \gamma_v \int_{\alpha}^{\beta} b_v(t)\, d\eta(t) - \int_{\alpha}^{\beta} \sum_{v=0}^{\infty} b_v(t)\gamma_v \, d\eta(t)$$

satisfies the inequality

$$|\sigma_n| \leqslant \mathrm{V}_{\alpha}^{\beta}\eta \cdot \sup_{n,\alpha\leqslant t\leqslant\beta} \Big| \sum_{v=n+1}^{\infty} b_v(t)\gamma_v \Big|$$

and because the series $\sum_{v=0}^{\infty} b_v(t)\gamma_v$ is uniformly convergent in $\langle \alpha, \beta \rangle$.

Now we shall prove that also

$$\sum_{n=0}^{\infty} \int_{\alpha}^{\beta} \Big[\sum_{v=0}^{n} b_v(t)\gamma_v \Big] d\eta_n(t) = \sum_{v=0}^{\infty} \gamma_v \int_{\alpha}^{\beta} b_v(t)\, d\Big(\sum_{n=v}^{\infty} \eta_n(t) \Big).$$

Let us consider the difference

$$\delta_p = \sum_{n=0}^{\infty} \int_{\alpha}^{\beta} \Big[\sum_{v=0}^{n} b_v(t)\gamma_v \Big] d\eta_n(t) - \sum_{v=0}^{p} \gamma_v \int_{\alpha}^{\beta} b_v(t)\, d\Big(\sum_{n=v}^{\infty} \eta_n(t) \Big);$$

hence

$$\delta_p = \sum_{n=0}^{\infty} \int_{\alpha}^{\beta} \Big[\sum_{v=0}^{n} b_v(t)\gamma_v \Big] d\eta_n(t) - \sum_{n=0}^{\infty} \int_{\alpha}^{\beta} \Big[\sum_{v=0}^{\inf(n,p)} b_v(t)\gamma_v \Big] d\eta_n(t)$$

$$= \sum_{n=p+1}^{\infty} \int_{\alpha}^{\beta} \Big[\sum_{v=p+1}^{n} b_v(t)\gamma_v \Big] d\eta_n(t).$$

By the uniform convergence of $B(t, \Gamma)$ in $\langle \alpha, \beta \rangle$, we have

$$\Big| \sum_{v=p+1}^{n} b_v(t)\gamma_v \Big| < \varepsilon$$

for large p and $n > p$ and hence

$$|\delta_p| \leqslant \varepsilon \sum_{n=p+1}^{\infty} \mathrm{V}_{\alpha}^{\beta}\eta_n.$$

Then from condition (ii) it follows that $\lim_{p\to\infty} \delta_p = 0$, which completes the proof.

THEOREM IX. *The general form of the linear (def. 14) functional* $F(\Gamma)$ $(\Gamma = \sum \gamma_v)$ *in the c-field* B_c^* *of the continuous method* B *of summation is*

(1) $$F(\Gamma) = \int_{t_0}^{T} B(t, \Gamma)\, d\varphi(t) + f(\Gamma),$$

where $\varphi(t)$ *is a function of bounded variation in the interval* $t_0 \leqslant t \leqslant T$, *and* $f(\Gamma)$ *satisfies the same conditions as in theorem* VIII.

Proof. The c-field B_c^* is equivalent to some closed subset in the Cartesian product $B_c^{**} \times C$, where C is a space of continuous functions in the interval $t_0 \leqslant t \leqslant T$. The functional may be extended to the whole space (see [1], p. 27, th. 1, p. 29 corollary and [3], p. 138), and a linear functional in a Cartesian product is the sum of linear functionals on each of its components ([3], p. 140). The form of the linear functional in C is well-known ([1], p. 59).

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