We may suppose that X is a subset of the Hilbert cube. Let $Z=\overline{X}$, $F_1=\overline{X}-X$ and $F_n=0$ for $n\geqslant 2$. Since X is locally compact, the set F_1 is closed. The sequence $\{F_n\}$ satisfying the assumptions of theorem (iii), the space X possesses the property (B), q. e. d.

Theorem (iv) can be deduced directly from (i). Namely, it is sufficient to pose $X_0=\overline{X}$ and $X_1=X$ in (i), and to denote by $\boldsymbol{F}$ the collection containing the set $\overline{X}-X+(x_0)$, where x_0 is a point of X, and all one-point sets (x), where $x\in X-(x_0)$.

By (iv) every open subset of a Euclidean space (or of the Hilbert cube) possesses the property (B).

LINEAR FUNCTIONALS ON DENJOY-INTEGRABLE FUNCTIONS

BY

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1. All the functions appearing throughout this paper are defined on an arbitrary but fixed closed interval $\langle a,b\rangle$.

Denote by (D) the linear space composed of the Denjoy-integrable functions $x=x(t)$, with the usual definition of addition and multiplication by real numbers. In this space we introduce a norm by the formula

$$\|x\|^*=\max_{a\leqslant s\leqslant b}\left|(D)\int_a^s x(t)\,dt\right|.$$

We consider two arts of convergence in (D). A sequence $\{x_n\}$ of elements of (D) will be called to be *$(*)$-convergent to x_0* if $\|x_n-x_0\|^*\to 0$ [1]; a sequence $\{x_n\}$ of elements of (D) will be called *η-convergent to x_0* if the sequence $(D)\int_a^s x_n(t)\,dt$ is

1° uniformly bounded,

2° asymptotically convergent to $(D)\int_a^s x_0(t)\,dt$,

3° convergent to $(D)\int_a^b x_0(t)\,dt$ for $s=b$.

A functional $F(x)$ defined in (D) is called *additive* if $F(\lambda x_1+\mu x_2)=\lambda F(x_1)+\mu F(x_2)$, where λ and μ are arbitrary numbers. An additive functional will be called *$(*)$-linear* or *η-linear* respectively if, given any sequence $\{x_n\}$ $(*)$-convergent or η-convergent to x_0 respectively, we have

$$\lim_{n\to\infty}F(x_n)=F(x_0).$$

The purpose of this paper is to characterize the $(*)$-linear and η-linear functionals in the space (D).

[1]) The space (D) normed by this formula is not complete.

2. We shall denote by (M) the linear space composed of the bounded and measurable functions $x=x(t)$. The norm being defined by the formula $\|x\|=\operatorname*{ess\,sup}_{a\leqslant t\leqslant b}|x(t)|$ [2]), (M) is a Banach space.

(C) will denote the linear subspace of (M) composed of the continuous functions and $(\hat{C})$ — the linear subset of (C) composed of the functions vanishing for $t=a$ and being indefinite Denjoy-integrals.

Theorem 1. The general form of the (*)-linear functionals in (D) is

$$F(x)=(D)\int_a^b x(t)\,h(t)\,dt, \tag{1}$$

where $h(t)$ is a function of bounded variation, continuous at the right, and such that $h(b)=0$; the norm of this functional is

$$\|F\|=\operatorname*{var}_{a\leqslant t\leqslant b} h(t).$$

Proof. Given any element $x=x(t)$ of (D), write

$$\hat{x}=\hat{x}(s)=(D)\int_a^s x(t)\,dt; \tag{2}$$

by this formula an equivalence (i. e. an one-to-one and isometrical mapping) between the spaces (D) and $(\hat{C})$ is defined; thus $\|x\|^*=\|\hat{x}\|$. Let $F(x)$ be any (*)-linear functional in (D). Writing $\hat{F}(x)=F(x)$ we get a linear functional in $(\hat{C})$ for which $\|F\|=\|\hat{F}\|$. By the well-known theorem of Banach-Hahn the functional $\hat{F}(x)$ may be extended to the whole of (C) without altering the norm. Hence we have by the Riesz theorem

$$\hat{F}(\hat{x})=\int_a^b \hat{x}(t)\,dg(t),$$

where $g(t)$ is a function of bounded variation; we may assume $g(t)$ to be continuous at the right and $g(b)=0$; the norm of the functional is

$$\|\hat{F}\|=\operatorname*{var}_{a\leqslant t\leqslant b} g(t).$$

[2]) i. e. $\|x\|=\inf_E[\sup_{x\epsilon E}|x(t)|]$, where E is any set of measure $b-a$.

If $\hat{x}\epsilon(\hat{C})$, we get integrating by parts,

$$\hat{F}(\hat{x})=-(D)\int_a^b x(t)\,g(t)\,dt.$$

Putting $h(t)=-g(t)$ we obtain the formula (1) as $\hat{F}(\hat{x})=F(x)$ for $\hat{x}\epsilon(\hat{C})$. Hence any (*)-linear functional in (D) is of the form (1).

To prove the converse, it suffices to remark that, given any linear functional $\hat{F}(y)$ in $(\hat{C})$, the formula $F(x)=\hat{F}(\hat{x})$ defines a (*)-linear functional in (D).

3. A sequence $\{x_n\}$ of elements of (M) will be termed *η-convergent to x_0* if

1^0 $\|x_n\|\leqslant k$, 2^0 $\lim_{n\to\infty}\text{as}\, x_n(t)=x_0(t)$ [3]) and 3^0 $x_n(b)\to x_0(b)$.

An additive functional $F(x)$ defined in (M) will be called *η-linear* if $F(x_n)\to F(x_0)$ for any sequence $\{x_n\}$ η-convergent to x_0. It follows easily by a theorem of Fichtenholz [4]) that the general form of the η-linear functionals in (M) is

$$F(x)=\int_a^b x(t)\,g(t)\,dt+Bx(b),$$

where $g(t)$ is a Lebesgue-integrable function.

A functional $F(x)$ defined in the linear subset $(\hat{C})$ of (M) will be termed *η-linear* if, given any sequence $\{x_n\}$ of elements of $(\hat{C})$ η-convergent to an element $x_0\epsilon(\hat{C})$, we have $F(x_n)\to F(x_0)$.

Lemma. Given any η-linear functional $\hat{F}(x)$ in $(\hat{C})$, there exists a η-linear functional $F(x)$ defined in (M) such that $F(x)=\hat{F}(x)$ for $x\epsilon(\hat{C})$.

Proof. Let x_0 be an arbitrary element of (M), $\{x_n\}$ any sequence of elements of $(\hat{C})$, η-convergent to x_0. The limit $\lim_{n\to\infty}\hat{F}(x_n)$ exists for, given two arbitrary sequences $\{p_n\}$ and $\{q_n\}$

[3]) $\lim_{n\to\infty}\text{as}\, x_n(t)$ denotes the asymptotical limit of the sequence $\{x_n(t)\}$.

[4]) G. Fichtenholz, *Sur les fonctionnelles linéaires continues au sens généralisé*, Recueil Mathématique 4 (1938), p. 193-214, especially p. 200.

of indices, the sequence $\{x_{p_n}-x_{q_n}\}$ is obviously η-convergent to 0; hence $\lim_{n\to\infty}[\hat{F}(x_{p_n})-\hat{F}(x_{q_n})]=\lim_{n\to\infty}\hat{F}(x_{p_n}-x_{q_n})=0$. We define

$$F(x_0)=\lim_{n\to\infty}\hat{F}(x_n).$$

To prove that $F(x)$ is η-linear in (M), consider a sequence $\{x_n\}$ of elements of (M) η-convergent to x_0 (vid. 1°-3°, p. 291). There exists a sequence $\{\hat{x}_n\}$ of elements of $(\hat{C})$ η-convergent to x_0 such that $\hat{F}(\hat{x}_n)\to F(x_0)$; for this sequence we have $\|\hat{x}_n\|\leqslant L$, $\lim_{n\to\infty}$ as $\hat{x}_n(t)=x_0(t)$ and $\hat{x}_n(b)\to x_0(b)$. Given any n, we can choose a sequence $\{x_{ni}\}$ such that

(i) $\qquad x_{ni}\,\epsilon\,(\hat{C}),\quad \|x_{ni}\|\leqslant 2K,\quad |x_{ni}(t)-x_n(t)|<1/n$

on a set of measure greater than $b-a-1/in$;

(ii) $\qquad x_{ni}(b)=x_n(b).$

Hence $\lim_{i\to\infty}\hat{F}(x_{ni})=F(x_n)$. Choose $m=m(n)$ so as to have

$$|\hat{F}(x_{nm})-F(x_n)|<1/n.$$

We observe easily that the sequence $\{x_{nm}-\hat{x}_n\}$ is η-convergent to 0. It follows that

$$\lim_{n\to\infty}[F(x_n)-F(x_0)]=$$
$$=\lim_{n\to\infty}[F(x_n)-F(x_{nm})]+\lim_{n\to\infty}[F(x_{nm})-F(\hat{x}_n)]+\lim_{n\to\infty}[F(\hat{x}_n)-F(x_0)]=$$
$$=\lim_{n\to\infty}[F(x_n)-\hat{F}(x_{nm})]+\lim_{n\to\infty}\hat{F}(x_{nm}-\hat{x}_n)+\lim_{n\to\infty}[\hat{F}(\hat{x}_n)-F(x_0)]=0.$$

Theorem 2. *The general form of the η-linear functionals in (D) is*

$$F(x)=(D)\int_a^b x(t)h(t)\,dt,$$

where $h(t)$ is an absolutely continuous function.

Proof. Given any η-linear functional $F(x)$ in (D), put using the operation (2)

$$F^*(\hat{x})=F(x).$$

$F^*(\hat{x})$ being an η-linear functional in $(\hat{C})$, there exists by the Lemma a functional F η-linear in (M), such that $F(y)=F^*(y)$ for $y\,\epsilon\,(\hat{C})$. Conversely, given any η-linear functional F in (M), the formula $F(x)=F^*(\hat{x})$ defines an η-linear functional in (D).

Since $F(y)=\int_a^b y(t)g(t)\,dt+By(b)$, where $g(t)$ is a Lebesgue-integrable function, we get for $\hat{x}\,\epsilon\,(\hat{C})$, integrating by parts,

$$F^*(\hat{x})=F(\hat{x})=\hat{x}(b)\hat{g}(b)-(D)\int_a^b x(t)\hat{g}(t)\,dt+B\hat{x}(b),$$

where $\hat{g}(t)=\int_a^t g(s)\,ds$. Hence

$$F(x)=\int_a^b x(t)\,[\hat{g}(b)+B-\hat{g}(t)]\,dt.$$