This problem is unsolved. It is known only that

(v) *If $2^{\aleph_0} < 2^{\aleph_1}$, every topological completely normal*[11] *space with the property (D) possesses also the property (I).*

Suppose that a completely normal space contains an enumerable dense subset X_0 and a non-enumerable isolated subset Y_0. For every set $Y \subset Y_0$ we have $\overline{Y}\cdot(Y_0 - Y) + Y\cdot\overline{(Y_0 - Y)} = 0$. Thus there exists an open set G_Y such that $Y \subset G_Y$ and $\overline{G}_Y\cdot(Y_0 - Y) = 0$. Let $X_Y = X_0 \cdot G_Y$. If $Y_1 \neq Y_2$, then $X_{Y_1} \neq X_{Y_2}$. The one-one mapping X_Y maps the class of all subsets of Y_0 in the class of all subsets of X_0 in contradiction with $2^{\aleph_0} < 2^{\aleph_1}$.

[11]) A space $\mathfrak{X}$ is called *completely normal* if for every two sets X_1, X_2 such that $\overline{X}_1\cdot X_2 + X_1\cdot\overline{X}_2 = 0$ there exists an open set G such that $X_1 \subset G$ and $\overline{G}\cdot X_2 = 0$. A space $\mathfrak{X}$ is completely normal if and only if every subspace $X \subset \mathfrak{X}$ is normal (see e. g. C. Kuratowski, op. cit., p. 130, Remarques).

REMARKS ON A PROBLEM OF BANACH

BY

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S. Banach has posed the following problem [1]):

When is it possible to define on a metric space X with a metric $\varrho(x_1, x_2)$ another metric $\varrho_1(x_1, x_2)$ such that

(1) *if* $\lim\limits_{n=\infty} \varrho(x_n, x) = 0$, *then* $\lim\limits_{n=\infty} \varrho_1(x_n, x) = 0$;

(2) *the metric space X_1 which we obtain from X by admitting the function $\varrho_1(x_1, x_2)$ as the metric is compact?*

It is easy to see that Banach's problem is equivalent to the question under what conditions a metric space X possesses the following property:

(B) *There exists a one-one continuous mapping f of X onto a compact metric space Y.*

It is clear that the geometrical image $\underset{xy}{E}[y = \varphi(x)]$ of an arbitrary real function $\varphi(x)$ $(0 \leqslant x \leqslant 1)$ possesses the property (B). The function f is then the projection on the x-axis.

W. Sierpiński has constructed a connected plain set S which is both F_σ and G_δ and which is the sum of an enumerable sequence $\{I_n\}$ of mutually disjoint simple arcs [2]). The set S does not possess the property (B). In fact, suppose that there exists a one-one continuous mapping f such that $f(S)$ is compact. Since S is connected, $f(S)$ would be a continuum. Since f is one-one, the continuum $f(S)$ would be the sum of the enumerable sequence $\{f(I_n)\}$ of mutually disjoint continuums, which is impossible [3]).

[1]) See this volume, p. 150, P26.

[2]) W. Sierpiński, *Sur quelques propriétés topologiques du plan*, Fundamenta Mathematicae 4 (1923), p. 5. I_n is the sum of the segment $x = 1/n$, $0 \leqslant y \leqslant 1$ and of the part of the circle $x^2 + y^2 = 1/n^2$, where either $x \leqslant 0$ or $y \leqslant 0$.

[3]) See W. Sierpiński, Tôhoku Mathematical Journal 13 (1918), p. 300, and F. Hausdorff, *Mengenlehre*, Berlin-Leipzig 1927, p. 162.

The two above examples show the difficulty of the characterizing of spaces with the property (B) by other topological properties of these spaces: on the one hand, there exist very singular spaces with the property (B) (e. g. geometrical images of non-measurable functions etc.); on the other hand, there exist very simple spaces without the property (B) (e. g. Sierpiński's set S).

It follows from a well-known theorem on semi-continuous decompositions [4]) that

(i) *A separable metric space X possesses the property (B) if and only if there exists a compact metric space X_0 such that*

1° X_0 contains a subset X_1 homeomorphic to X;

2° there exists a semi-continuous decomposition [5]) $\boldsymbol{F}$ of the space X_0 such that for every $F\epsilon\boldsymbol{F}$ the set X_1F contains exactly one point.

Suppose that the conditions 1° and 2° are fulfilled and let h denote a homeomorphism of X onto X_1. Since the decomposition $\boldsymbol{F}$ is semi-continuous, there exist a compact space Y and a continuous mapping g of X_0 onto Y such that $g^{-1}(y)\epsilon\boldsymbol{F}$ for every $y\epsilon Y$ [4]). The continuous mapping $f=gh$ is one-one on account of 2° and maps X onto Y. Thus the space X possesses the property (B).

Suppose now that the space X possesses the property (B), i. e. that there exist a compact space Y and a one-one continuous mapping f of X onto Y. We may suppose that X is a subset of the Hilbert cube H. Let $X_0=H\times Y$, $X_1=\underset{xy}{E}[y=f(x)]$, and let $\boldsymbol{F}$ be the collection of all sets $H\times(y)$ where $y\epsilon Y$. $\boldsymbol{F}$ is a semi-continuous decomposition of the compact space X_0. Since f is one-one

[4]) See G. T. Whyburn, *Analytic Topology*, New York 1942, p. 126, theorem (3.4).

[5]) A *semi-continuous decomposition* of X_0 is a collection $\boldsymbol{F}$ of mutually disjoint closed subsets of X_0 such that:

1. X_0 is the sum of all sets $F\epsilon\boldsymbol{F}$;

2. for every $F\epsilon\boldsymbol{F}$ and for every sequence $F_n\epsilon\boldsymbol{F}$, if $F\cdot\underset{n=\infty}{\mathrm{Li}}F_n\neq0$, then $\underset{n=\infty}{\mathrm{Ls}}F_n\subset F$.

$\underset{n=\infty}{\mathrm{Li}}F_n$ and $\underset{n=\infty}{\mathrm{Ls}}F_n$ denote respectively the topological limes inferior and the topological limes superior of the sequence $\{F_n\}$. See C. Kuratowski, *Topologie I* (new edition), Monografie Matematyczne, Warszawa-Wrocław 1948, p. 241 and 243.

and $f(X)=Y$, the set $X_1(H\times(y))$ contains exactly one point for any $y\epsilon Y$. f being continuous, the set $X_1\subset X_0$ is a homeomorph of X. Thus the conditions 1° and 2° are satisfied and theorem (i) is proved.

It follows from this proof that the condition 1° of (i) can be replaced by the condition: *X_0 contains a subset X_1 which is a one-one and continuous image of X.*

(ii) *Let $\{F_n\}$ be a sequence of mutually disjoint closed subsets of a compact metric space Z. If for every integer m and for every subsequence $\{F_{m_n}\}$*

$$(*)\qquad F_m\cdot\underset{n=\infty}{\mathrm{Li}}F_{m_n}\neq0 \quad \textit{implies} \quad \underset{n=\infty}{\mathrm{Ls}}F_{m_n}\subset F_m \text{ [5]},$$

then the space $X=Z-\sum_{n=1}^{\infty}F_n$ possesses the property (B).

If X is finite, theorem (ii) is obviously true. Suppose that X is infinite. Let $X_0=\overline{X}$. Since X is dense in X_0, for every n there exists a point $x_n\epsilon X$ such that [6]) $\varrho(x_n,X_0F_n)<1/n$, $x_i\neq x_j$ for $i\neq j$. On account of $(*)$ the collection $\boldsymbol{F}$ of all sets $X_0F_n+(x_n)$ and of all one-point sets (x) where $x\epsilon X-\sum_{n=1}^{\infty}(x_n)$, is a semi-continuous decomposition of X_0. The sets X, X_0 and $X_1=X$, and the decomposition $\boldsymbol{F}$ satisfying the conditions 1° and 2°, the space X possesses the property (B), q. e. d.

(iii) *If $\{F_n\}$ is a sequence of mutually disjoint closed subsets of a compact metric space Z such that [7]) $\delta(F_n)\to0$, the space $X=Z-\sum_{n=1}^{\infty}F_n$ possesses the property (B).*

In fact, the sequence $\{F_n\}$ satisfies the condition $(*)$ of theorem (ii).

(iv) *Every locally compact separable space X possesses the property (B).*

[6]) $\varrho(x,A)$ denotes the lower bound of distances between x and any point of A. If $A=0$, then $\varrho(x,A)=0$.

[7]) $\delta(F)$ denotes the *diameter* of F, i. e. the upper bound of distances between any two points of F. If $F=0$, then $\delta(F)=0$.

We may suppose that X is a subset of the Hilbert cube. Let $Z=\overline{X}$, $F_1=\overline{X}-X$ and $F_n=0$ for $n\geqslant 2$. Since X is locally compact, the set F_1 is closed. The sequence $\{F_n\}$ satisfying the assumptions of theorem (iii), the space X possesses the property (B), q. e. d.

Theorem (iv) can be deduced directly from (i). Namely, it is sufficient to pose $X_0=\overline{X}$ and $X_1=X$ in (i), and to denote by $\boldsymbol{F}$ the collection containing the set $\overline{X}-X+(x_0)$, where x_0 is a point of X, and all one-point sets (x), where $x\epsilon X-(x_0)$.

By (iv) every open subset of a Euclidean space (or of the Hilbert cube) possesses the property (B).

LINEAR FUNCTIONALS ON DENJOY-INTEGRABLE FUNCTIONS

BY

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1. All the functions appearing throughout this paper are defined on an arbitrary but fixed closed interval $\langle a,b\rangle$.

Denote by (D) the linear space composed of the Denjoy-integrable functions $x=x(t)$, with the usual definition of addition and multiplication by real numbers. In this space we introduce a norm by the formula

$$\|x\|^*=\max_{a\leqslant s\leqslant b}\left|(D)\int_a^s x(t)\,dt\right|.$$

We consider two arts of convergence in (D). A sequence $\{x_n\}$ of elements of (D) will be called to be *$(*)$-convergent to x_0* if $\|x_n-x_0\|^*\to 0$ [1]; a sequence $\{x_n\}$ of elements of (D) will be called *η-convergent to x_0* if the sequence $(D)\int_a^s x_n(t)\,dt$ is

1° uniformly bounded,

2° asymptotically convergent to $(D)\int_a^s x_0(t)\,dt$,

3° convergent to $(D)\int_a^b x_0(t)\,dt$ for $s=b$.

A functional $F(x)$ defined in (D) is called *additive* if $F(\lambda x_1+\mu x_2)=\lambda F(x_1)+\mu F(x_2)$, where λ and μ are arbitrary numbers. An additive functional will be called *$(*)$-linear* or *η-linear* respectively if, given any sequence $\{x_n\}$ $(*)$-convergent or η-convergent to x_0 respectively, we have

$$\lim_{n\to\infty}F(x_n)=F(x_0).$$

The purpose of this paper is to characterize the $(*)$-linear and η-linear functionals in the space (D).

[1]) The space (D) normed by this formula is not complete.