ON A PROBLEM OF MARCZEWSKI

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Let $m(A)$ be a non-negative completely additive set function defined on a Borel field K of some space X, with $m(X)=1$. The sets of K will be called *measurable* below, and measurability of functions will be with reference to this field. Let $f(x)$ and $g(x)$ be real numerically valued measurable functions defined for $x\epsilon X$. If they satisfy the equation

$$(1)\qquad m\{f(x)\epsilon F, g(x)\epsilon G\} = m\{f(x)\epsilon F\}\cdot m\{g(x)\epsilon G\}$$

for all linear intervals F, G (and therefore for all linear Borel sets F, G) the functions are said to be *S-independent*. If (1) holds for all F and G for which the sets involved are measurable, the functions are said to be *K-independent*. Clearly K-independence implies S-independence.

Hartman[1]) has proved that S-independence implies K-independence if $m(A)$ is Lebesgue measure.

Marczewski[2]) has proposed the problem „Does S-independence always imply K-independence?".

The following example shows that the answer is „No"[3]).

Let X be the unit square $0\leqslant\xi,\eta\leqslant 1$. In the following $|L|_i$ will denote the i-dimensional Lebesgue measure of the set L. Let $\hat{F}$ be a linear set in the interval (0,1) of outer and inner one dimensional Lebesgue measures 1 and 0 respectively, and let F be the plane set $\{\xi\epsilon\hat{F}, 0\leqslant\eta\leqslant 1\}$. Then F has outer and inner two-dimensional Lebesgue measures 1 and 0 respectively. The field K is defined as the field of all sets of the form

$$(2)\qquad A = L_1F + L_2(X-F)$$

where L_1 and L_2 are (two-dimensional) Lebesgue measurable sets in the unit square. Let $\varphi(\xi,\eta)$ be a numerically valued real Lebesgue measurable function defined in the unit square, with $0\leqslant\varphi\leqslant 1$, and define $m(A)$ by

$$(3)\qquad m(A) = \iint_{L_1}\varphi(\xi,\eta)\,d\xi\,d\eta + \iint_{L_2}[1-\varphi(\xi,\eta)]\,d\xi\,d\eta.$$

Since A in (2) determines L_1 and L_2 uniquely, neglecting sets of Lebesgue measure 0, (3) defines $m(A)$ uniquely. This set function is defined on the field K, is completely additive, and if G is Lebesgue measurable and in the unit square, the equality $G = GF + G(X-F)$ shows that G is in K and that

$$(4)\qquad m(G) = \iint_G 1\,d\xi\,d\eta = |G|_2.$$

The equality $F = XF$ shows that F is in K and that

$$(5)\qquad m(F) = \int_0^1\!\!\int_0^1 \varphi(\xi,\eta)\,d\xi\,d\eta.$$

The two functions $f=\xi$ and $g=\eta$ are measurable and S-independent because $m(A)$ is an extension of Lebesgue measure in the unit square. We shall show that these functions are not K-independent unless φ has a very special character. In fact, if they were K-independent, and if G were a set of the form $\{\eta\epsilon\hat{G}\}$ where $\hat{G}$ is a linear Lebesgue measurable set of the interval (0, 1), we should have from (1)

$$(6)\qquad m\{\xi\epsilon\hat{F}, \eta\epsilon\hat{G}\} = m\{\xi\epsilon\hat{F}\}\cdot m\{\eta\epsilon\hat{G}\},$$

that is

$$(7)\qquad m(FG) = m(F)\cdot m(G),$$

which becomes

$$(8)\qquad \int_{\hat{G}} d\eta \int_0^1 \varphi(\xi,\eta)\,d\xi = \int_0^1\!\!\int_0^1 \varphi(\xi,\eta)\,d\xi\,d\eta\cdot|G|_2 = |\hat{G}|_1\int_0^1\!\!\int_0^1\varphi(\xi,\eta)\,d\xi\,d\eta.$$

If this is true for all $\hat{G}$, $\int_0^1\varphi(\xi,\eta)\,d\xi = \text{const.}$ for almost all η; any choice of φ not satisfying this condition gives a measure $m(A)$ in terms of which ξ and η are not K-independent although they are S-independent.

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[1]) This volume, p. 19-22.

[2]) This volume, p. 29-30, Problem **P3**.

[3]) Cf. also **P3**, **R2**, this volume p. 148, and the preceding paper of B. Jessen.