ON TWO NOTIONS OF INDEPENDENT FUNCTIONS

(From a Letter to E. Marczewski)

BY

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Your problem regarding independence of random variables is related to a question on conditional mean-values which Mr Sparre Andersen and I have had occasion to consider. It seems to be answered (negatively) by the following example.

Let $\mu$, $\mu^*$, and $\mu^*$ denote the Lebesgue measure and the exterior and interior Lebesgue measures on $I$: $0 \leq t < 1$, and let $\emptyset$ denote the system of all measurable sets in $I$. Let $Q$ be a set with $\mu(Q) = 1$, $\mu^*(Q) = 0$. Then if $\emptyset$ denotes the system of all sets of the form

$A = BQ \cup C(I-C), \quad B \in \emptyset, \quad C \in \emptyset$,

there are defined by

$\mu_1(A) = \mu(B), \quad \mu_2(A) = \mu(C)$

two measures in $I$ with domain $\emptyset$, both of which are extensions of $\mu$. We have $\mu_1(Q) = 1$, $\mu_2(Q) = 0$.

Consider now the square $S$: $0 \leq t_1 < 1$, $0 \leq t_2 < 1$, and let superscripts (1) and (2) indicate measures or sets on $0 \leq t_1 < 1$ and $0 \leq t_2 < 1$ respectively. Then the two product-measures

$\nu = (\mu^1, \mu_2^{(1)})$, \quad $\nu_2 = (\mu^2, \mu_1^{(2)})$,

and hence also $\lambda = \nu_1 \cup \nu_2$, are extensions of the measure $(\mu^1, \mu^2)$ in $S$.

1) See PS, this volume, p. 29-30; cf. also J. L. Doob, On a problem of Marcinkiewicz, this fascicle, p. 216-217.