ON TWO NOTIONS OF INDEPENDENT FUNCTIONS

(FROM A LETTER TO E. MARCZEWSKI)

BY

B. JESSEN (COPENHAGEN)

Your problem regarding independence of random variables [1]) is related to a question on conditional mean-values which Mr Sparre Andersen and I have had occasion to consider [2]). It seems to be answered (negatively) by the following example.

Let μ, μ^* and μ_* denote the Lebesgue measure and the exterior and interior Lebesgue measures on I: $0 \leqslant t < 1$, and let $\mathfrak{F}$ denote the system of all measurable sets in I. Let Q be a set with $\mu^*(Q)=1$, $\mu_*(Q)=0$. Then if $\mathfrak{G}$ denotes the system of all sets of the form

$$A = BQ + C(I-Q), \qquad B \epsilon \mathfrak{F}, \qquad C \epsilon \mathfrak{F},$$

there are defined by

$$\mu_1(A) = \mu(B), \qquad \mu_2(A) = \mu(C)$$

two measures in I with domain $\mathfrak{G}$, both of which are extensions of μ. We have $\mu_1(Q)=1$, $\mu_2(Q)=0$.

Consider now the square S: $0 \leqslant t_1 < 1$, $0 \leqslant t_2 < 1$, and let superscripts (1) and (2) indicate measures or sets on $0 \leqslant t_1 < 1$ and $0 \leqslant t_2 < 1$ respectively. Then the two product-measures

$$\nu_1 = (\mu_1^{(1)}, \mu_2^{(2)}), \qquad \nu_2 = (\mu_2^{(1)}, \mu_1^{(2)}),$$

and hence also $\lambda = {}^1/_2(\nu_1 + \nu_2)$, are extensions of the measure $(\mu^{(1)}, \mu^{(2)})$ in S.

[1]) See **P3**, this volume, p. 29-30; cf. also J. L. Doob, *On a problem of Marczewski*, this fascicle, p. 216-217.

[2]) E. Sparre Andersen and B. Jessen, *Some limit theorems on integrals in an abstract set*, Danske Vid. Selsk. Mat.-Fys. Meddelelser, vol 22, 14 (1946), p. 25 (footnote).

But λ is not a product-measure, since

$$\lambda\big((I^{(1)}, Q^{(2)})\big) = {}^1/_2(0+1) = {}^1/_2,$$

$$\lambda\big((Q^{(1)}, I^{(2)})\big) = {}^1/_2(1+0) = {}^1/_2,$$

$$\lambda\big((Q^{(1)}, Q^{(2)})\big) = {}^1/_2(0+0) \neq {}^1/_4.$$

Hence the functions $f_1(t_1, t_2) = t_1$ and $f_2(t_1, t_2) = t_2$ are independent in Steinhaus's but not in Kolmogoroff's sense.

January 31, 1948.