CONCERNING THE EULER CHARACTERISTIC OF NORMAL SPACES

BY

K. BORSUK (WARSAW)

Let A be a normal space [1]) whose all Betti numbers $p^k(A)$ [2]) are finite and vanish for k above a certain value. The number

$$\chi(A)=\sum_{k=0}^{\infty}(-1)^k\cdot p^k(A) \tag{1}$$

is called the *Euler characteristic* of A. If A is a polytope decomposed into a simplicial complex containing a_k different k-dimensional simplexes then the Euler characteristic of A can be expressed by the following Euler-Poïncaré formula [3])

$$\chi(A)=\sum_{k=0}^{\infty}(-1)^k a_k.$$

By a *homologically regular decomposition* of A we mean a finite sequence

$$\mathfrak{A}=\{A_1, A_2, \ldots, A_m\}$$

of closed subsets of A such that

$$A=A_1+A_2+\ldots+A_m$$

and that for every sequence $i_1, i_2, \ldots, i_\nu$ of indices $\leqslant m$ the set $A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_\nu}$ is either empty or acyclic [4]). In particular, every one of the sets $A_1, A_2, \ldots, A_m$ is either empty or acyclic.

[1]) A topological space is said to be *normal*, whenever for every two disjoint closed sets there exist disjoint neighbourhoods.

[2]) $p^k(A)$ denotes the ordinary k-dimensional Betti number of A, that is the rank of the k-dimensional discrete homology group of A (in the sense of E. Čech, *Théorie générale de l'homologie dans un espace quelconque*, Fundamenta Mathematicae 19 (1932), p. 147-183) with rational coefficients.

[3]) See, for instance, S. Lefschetz, *Algebraic Topology*, American Mathematical Society Colloquium Publications 27, New York 1942, p. 104.

[4]) A normal space A is said to be acyclic, if $p^0(A)=1$ and $p^k(A)=0$ for every $k=1, 2, \ldots$

Let $\beta_k(\mathfrak{A})$ denote, for $k=1,2,\ldots,m$, the number of all different increasing sequences $i_1, i_2, \ldots, i_k$ of indices $\leqslant m$ for which $A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_k}\neq 0$. Using the notion of the nerve of a covering, introduced by Alexandroff [5]), we can also define the number $\beta_k(\mathfrak{A})$ as the number of all $(k-1)$-dimensional simplexes of the nerve $N(\mathfrak{A})$ of the decomposition $\mathfrak{A}$.

Theorem. Let $\mathfrak{A}=\{A_1, A_2, \ldots, A_m\}$ be a homologically regular decomposition of a normal space A. Then:

1⁰ $p^k(A)<\infty$ *for every* $k=0,1,\ldots,$

2⁰ $p^k(A)=0$ *for every* $k\geqslant m,$

3⁰ $\chi(A)=\sum_{k=1}^{m}(-1)^{k+1}\beta_k(\mathfrak{A}).$

Evidently, the statement 3⁰ may be also formulated in the following manner:

3⁰′ $\chi(A)=\chi(N(\mathfrak{A})).$

Proof. If $A=0$ then $p^k(A)=\beta_k(\mathfrak{A})=\chi(A)=0$ for every $k=0,1,\ldots,$ and the theorem is true. Consequently we can further assume that $A\neq 0$. Hence at least one of the sets $A_1, A_2, \ldots, A_m$ is not empty; we can assume that $A_m\neq 0$.

We apply the induction on the number m. For $m=1$ the space $A=A_m$ is acyclic and we have $p^0(A)=1$ and $p^k(A)=0$ for every $k>0$ and $\chi(A)=\beta_1(\mathfrak{A})=1$. Hence for $m=1$ the theorem is true.

Now let $m=m_0+1>1$ and assume the validity of the theorem for $m=m_0$. Setting

$$A'=A_1+A_2+\ldots+A_{m_0}, \qquad A''=A_m$$

we have

$$A=A'+A''. \tag{2}$$

It is clear that $\mathfrak{A}'=\{A_1, A_2, \ldots, A_{m_0}\}$ constitutes a homologically regular decomposition of A' and

$$\mathfrak{A}^*=\{A_1\cdot A_m, A_2\cdot A_m, \ldots, A_{m_0}\cdot A_m\}$$

[5]) The *nerve* of the decomposition $\mathfrak{A}=\{A_1, A_2, \ldots, A_m\}$ is a simplicial complex $N(\mathfrak{A})$ having the not empty of the sets $A_1, A_2, \ldots, A_m$ as vertices and the systems $(A_{i_1}, A_{i_2}, \ldots, A_{i_k})$ with $A_{i_1}, A_{i_2}, \ldots, A_{i_k}\neq 0$ as simplexes. See P. Alexandroff, *Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung*, Mathematische Annalen 98 (1928), p. 634.

— a homologically regular decomposition of $A'\cdot A''$. By the hypothesis of the induction we have

(3) $p^k(A')<\infty$ and $p^k(A'\cdot A'')<\infty$ for every $k=0,1,\ldots,$

(4) $p^k(A')=p^k(A'\cdot A'')=0$ for every $k\geqslant m_0$,

(5) $\chi(A')=\sum_{k=1}^{m_0}(-1)^{k+1}\beta_k(\mathfrak{A}')$, $\chi(A'\cdot A'')=\sum_{k=1}^{m_0}(-1)^{k+1}\beta_k(\mathfrak{A}^*)$.

Now let us apply the known formula of Mayer-Vietoris-Čech [6])

(6) $p^k(A'+A'')+p^k(A'\cdot A'')=p^k(A')+p^k(A'')+$
$+\pi^k(A',A'')+\pi^{k-1}(A',A'')$,

where $\pi^l(A',A'')$ denotes (for $l\geqslant 0$) the rank of the subgroup of the l-dimensional homology group of $A'\cdot A''$ (with rational coefficients) constituted by all classes of cycles bounding in both sets A' and A'', and $\pi^{-1}(A',A'')=0$.

Obviously, it is

$$\pi^l(A',A'')\leqslant p^l(A'\cdot A'') \quad \text{for every } l=0,1,\ldots$$

By (3) and (4) we infer:

(7) $\pi^l(A',A'')<\infty$ for every $l=-1,0,1,\ldots,$

(8) $\pi^l(A',A'')=0$ for every $l\geqslant m_0$.

From (2), (3), (6) and (7) we conclude that $p^k(A)=p^k(A'+A'')$ is finite for every $k=0,1,\ldots$, i. e. the statement 1⁰ is established. From (2), (4), (6) and (8) it follows that

$$p^k(A)=p^k(A'+A'')=p^k(A'')=0 \quad \text{for every } k\geqslant m.$$

Hence the statement 2⁰ is established.

Now let us observe that $\beta_k(\mathfrak{A}^*)$ (for $k=1,2,\ldots,m_0$) denotes the number of all different increasing sequences $i_1,i_2,\ldots,i_k$ with natural terms $\leqslant m_0$ such that

$$(A_{i_1}\cdot A_m)\cdot(A_{i_2}\cdot A_m)\cdot\ldots\cdot(A_{i_k}\cdot A_m)\neq 0.$$

To each such sequence corresponds the increasing sequence $i_1,i_2,\ldots,i_k,m$ such that

$$A_{i_1}\cdot A_{i_2}\cdot\ldots\cdot A_{i_k}\cdot A_m\neq 0.$$

[6]) E. Čech, loco cit., p. 178.

Obviously, if we adjoin the collection of all sequences of this last form to the collection of all increasing sequences $j_1,j_2,\ldots,j_{k+1}$ with natural terms not greather than m_0 and with

$$A_{j_1}\cdot A_{j_2}\cdot\ldots\cdot A_{j_{k+1}}\neq 0,$$

then we obtain all increasing sequences $j_1,j_2,\ldots,j_{k+1}$ with natural terms $\leqslant m$ and with $A_{j_1}\cdot A_{j_2}\cdot\ldots\cdot A_{j_{k+1}}\neq 0$. It follows

(9) $\beta_k(\mathfrak{A}^*)=\beta_{k+1}(\mathfrak{A})-\beta_{k+1}(\mathfrak{A}')$ for every $k=1,2,\ldots,m_0$.

Furthermore, immediately from the definition of $\beta_1(\mathfrak{A})$ we infer that

(10) $\beta_1(\mathfrak{A}')=\beta_1(\mathfrak{A})-1$.

Applying the formulas (1), (5), (6), (8), (9) and (10) we have:

$$\chi(A)=\chi(A'+A'')=\sum_{k=0}^{\infty}(-1)^k p^k(A'+A'')=\sum_{k=0}^{\infty}(-1)^k p^k(A')+\sum_{k=0}^{\infty}(-1)^k p^k(A'')+$$
$$-\sum_{k=0}^{\infty}(-1)^k p^k(A'\cdot A'')+\sum_{k=0}^{\infty}(-1)^k\pi^k(A',A'')+\sum_{k=0}^{\infty}(-1)^k\pi^{k-1}(A',A'')=$$
$$=\chi(A')+1-\chi(A'\cdot A'')+\pi^{-1}(A',A'')=\sum_{k=1}^{m_0}(-1)^{k+1}\beta_k(\mathfrak{A}')+1-\sum_{k=1}^{m_0}(-1)^{k+1}\beta_k(\mathfrak{A}^*)=$$
$$=\sum_{k=1}^{m_0}(-1)^{k+1}\beta_k(\mathfrak{A}')+1-\sum_{k=1}^{m_0}(-1)^{k+1}\beta_{k+1}(\mathfrak{A})+\sum_{k=1}^{m_0}(-1)^{k+1}\beta_{k+1}(\mathfrak{A}')=$$
$$=\beta_1(\mathfrak{A}')+1+\sum_{k=2}^{m_0+1}(-1)^{k+1}\beta_k(\mathfrak{A})=\sum_{k=1}^{m}(-1)^{k+1}\beta_k(\mathfrak{A}).$$

This concludes the proof of the statement 3⁰ and consequently, by induction, establishes the theorem.