Now $a_m^2 F(m) - b_m^2 F(-m)$ is an *integer*. But for $-m < x < m$:

$$0 < a_m b_m e^x f(x) < \frac{a_m^2 m^{2n}}{n!},$$

so that the integral in (*) is *positive, but arbitrarily small for n sufficiently large*. Thus (*) is false, hence e^m (for $m = 1, 2, 3, \ldots$) is irrational.

CONCERNING THE SYMMETRIC DIFFERENCE IN THE THEORY OF SETS AND IN BOOLEAN ALGEBRAS

BY

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The set $(A - B) + (B - A) = A + B - AB$, i. e. the set of all points which belong to one and only one of the sets A and B, is called the symmetric difference of A and B and will denote in this note by $A \dot{-} B$. The classical books on the General Theory of Sets by Hausdorff, Sierpiński etc. did not treat the symmetric difference [1]), but this operation has proved useful in a number of recent papers. It has been applied especially to two fields: 1° in *measure theory*, the distance of two sets can be defined as the measure of their symmetric difference (Nikodym and Aronszajn) [2]), 2° the class of all subsets of a set (or, more generally, any Boolean algebra) forms a ring in the *algebraic* sense under the operations symmetric difference and multiplication [3]); in particular, symmetric difference is a group operation.

Some investigations in the measure theory suggest a partial analogy between the symmetric difference of sets and the absolute value of the difference of numbers. This idea leads (§ 1) to a new formulation of Kantorovitch-Livenson definition of quasi-analytical operations [4]).

In § 2 I show that the symmetric difference is the only group operation among the binary quasi-analytical operations.

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[1]) Hausdorff treats the symmetric difference only in the last edition of his book (F. Hausdorff, *Mengenlehre*, Berlin-Leipzig 1935, Ergänzungen, p. 276-278).

[2]) See e. g. O. Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, Fundamenta Mathematicae 15 (1930), p. 131-179, especially p. 137, Definition 4.

[3]) See e. g. G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications 25, New York 1940, p. 96.

[4]) L. Kantorovitch and E. Livenson, *Memoir on the Analytical Operations and Projective Sets I*, Fundamenta Mathematicae 18 (1932), p. 214-279, especially p. 239, Definition 1'.

This may be treated as an answer to a question of Ulam, posed some times ago[5]). I due to Helson some modifications of this paper[6]).

§ 1. Analogy between the symmetric difference of sets and the absolute value of the difference of numbers. We state the following inclusions[7]) as obvious:

(1) $$\sum_\xi A_\xi \dot{-} \sum_\xi B_\xi \subset \sum_\xi (A_\xi \dot{-} B_\xi),$$

(2) $$\prod_\xi A_\xi \dot{-} \prod_\xi B_\xi \subset \sum_\xi (A_\xi \dot{-} B_\xi),$$

(3) $$(A_1 \dot{-} A_2) \dot{-} (B_1 \dot{-} B_2) \subset \sum_{j=1}^{2} (A_j \dot{-} B_j).$$

The question presents itself: which set operations $\Phi(A_1, A_2, \ldots)$ fulfil the analogous condition

(4) $$\Phi(A_1, A_2, \ldots) \dot{-} \Phi(B_1, B_2, \ldots) \subset \sum_\xi (A_\xi \dot{-} B_\xi)$$

for all sequences of sets $\{A_\xi\}$ and $\{B_\xi\}$?

Kantorovitch and Livenson call a set operation *quasi-analytical* whenever the relations

$$x \in \Phi(A_1, A_2, \ldots) \quad \text{and} \quad x \text{ non} \in \Phi(B_1, B_2, \ldots)$$

imply that for some ξ

$$x \in A_\xi \text{ and } x \text{ non} \in B_\xi \quad \text{or} \quad x \text{ non} \in A_\xi \text{ and } x \in B_\xi \text{ }^{8)}.$$

It is easy to see that the condition (4) is necessary and sufficient for a set operation $\Phi(E_1, E_2, \ldots)$ to be quasi-analytical.

We suggest that the condition (4) is more adequate than the primitive definition because it is applicable to Boolean algebras, inasmuch as it does not involve elements of sets.

The following propositions are also suggested by the analogy under consideration and are easy to prove:

(5) $$(A \dot{-} B) + (B \dot{-} C) \supset A \dot{-} C \text{ }^{7)},$$

[5]) See also the subsequent paper of Henry Helson, p. 203-205.

[6]) presented to the Warsaw Scientific Society, on June 25, 1948.

[7]) Cf. F. Hausdorff, op. cit., p. 276, (1) and (3).

[8]) E. g. the operation $\Phi(A,B) = (A+B)E_1 + ABE_2$, where E_1 and E_2 are fixed sets, is quasi-analytical.

(6) $\lim\limits_{n\to\infty} A_n = A$ *if and only if* $\lim\limits_{n\to\infty} (A \dot{-} A_n) = 0$ [9]),

(7) *the sequence* A_n *is convergent if and only if*

$$\lim_{m,n\to\infty} (A_m \dot{-} A_n) = 0 \text{ }^{10)},$$

(8) *every quasi-analytical operation* $\Phi(E_1, E_2, \ldots)$ *is continuous with respect to each variable* [9]).

§ 2. Symmetric difference as the only quasi-analytical group operation. Let $\boldsymbol{B}$ be a Boolean algebra with the operations $+$, $\cdot$ and $-$. We set as before

$$A \dot{-} B = (A - B) + (B - A)$$

and we denote by 0 the „empty" element and by X the „universal" element of $\boldsymbol{B}$.

For each $E \in \boldsymbol{B}$ we set $E^0 = X - E$ and $E^1 = E$. Consequently,

(9) $$X^i \dot{-} E = E^{1-i}$$

(10) $$E^i E^{1-i} = 0 \qquad \text{for} \quad E \in \boldsymbol{B}, \quad i = 0, 1,$$

(11) $$X = \sum_{i=0}^{1} \sum_{j=0}^{1} A^i B^j \quad \text{for} \quad i = 0, 1, \quad j = 0, 1.$$

The quasi-analyticity of an operation $E = \Phi(E_1, E_2)$ (where $E, E_1, E_2 \in \boldsymbol{B}$) is defined by the condition (4).

Lemma 1. *Each binary quasi-analytical operation* $\Phi(A, B)$ *is completely determined by the four elements of* $\boldsymbol{B}$*:*

$$\Phi(0,0), \quad \Phi(0,X), \quad \Phi(X,0) \quad \textit{and} \quad \Phi(X,X).$$

More precisely

(12) $$\Phi(A,B) = \sum_{i=0}^{1} \sum_{j=0}^{1} A^i B^j \Phi(X^i X^j).$$

Proof. By (11) we have

$$\Phi(A,B) = \sum_{i=0}^{1} \sum_{j=0}^{1} A^i B^j \Phi(A,B).$$

[9]) where the convergence is meant in the sense of the General Theory of Sets.

[10]) $\lim\limits_{m,n\to\infty} Z_{m,n} = 0$ means that for each sequence $[m_j, n_j]$ of ordered pairs of natural numbers such that $m_j \to +\infty$ and $n_j \to \infty$, we have $Z_{m_1,n_1} \cdot Z_{m_2,n_2} \cdot \ldots = 0$.

Consequently, it suffices to prove then

$$A^iB^j\Phi(A,B)=A^iB^j\Phi(X^i,X^j)$$

or, in other terms,

$$(13)\qquad A^iB^j\Phi(A,B)\dot{-}A^iB^j\Phi(X^i,X^j)=0.$$

By the quasi-analyticity of Φ and the relations (9) and (10) we have

$$A^iB^j\Phi(A,B)\dot{-}A^iB^j\Phi(X^iX^j)=$$
$$=A^iB^j[\Phi(X^i,X^j)\dot{-}\Phi(A,B)]\subset A^iB^j[(X^i\dot{-}A)+(X^j\dot{-}B)]=$$
$$=A^iB^jA^{1-i}+A^iB^jB^{1-j}=0,$$

whence the equality (13).

By Lemma 1 we obtain immediately the

Lemma 2. If for a quasi-analytical operation $\Phi(A,B)$ holds

$$\Phi(0,0)=\Phi(X,X)=0 \quad \text{and} \quad \Phi(0,X)=\Phi(X,0)=X,$$

then we have identically $\Phi(A,B)=A\dot{-}B$.

*Theorem. Suppose a binary quasi-analytical operation $\circ$ is defined in a family **F** of elements of a Boolean algebra **B** such that the „empty" element 0 and the „universal" element X of **B** belong to **F** and that **F** is a group with respect to $\circ$, with „empty" element as unit. Then*

$$(14)\qquad A\circ B=A\dot{-}B \quad \text{for each } A,B\in\boldsymbol{F}.$$

Proof. By group properties,

$$(15)\qquad 0\circ 0=0,\quad 0\circ X=X \quad\text{and}\quad X\circ 0=X$$

By formula (12) of Lemma 1 and by (15) we have

$$A\circ X\supset(X-A)X(0\circ X)=X-A \quad \text{for} \quad A\in\boldsymbol{F},$$

whence $A\circ X\neq 0$ for $A\neq X$. On the other hand, there is by group properties a $B\in\boldsymbol{F}$ such that $B\circ X=0$, whence

$$(16)\qquad X\circ X=0.$$

By (15), (16) and Lemma 2 we obtain (14), q. e. d.

ON THE SYMMETRIC DIFFERENCE OF SETS AS A GROUP OPERATION

BY

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If M is a set of elements $a,b,\ldots$ and $\boldsymbol{M}$ the field of all subsets $A,B,\ldots$ of M, then $\boldsymbol{M}$ is a group under the point-set operation symmetric difference:

$$A\dot{-}B=(A-B)+(B-A).$$

Evidently the group is completely determined by the power of M, and is commutative [1].

Suppose $\boldsymbol{M}$ is a group with respect to some binary operation $\circ$. S. Ulam has asked what further conditions can be imposed on $\circ$ to characterize the operation as symmetric difference. Marczewski has shown [2] that quasi-analyticity is a sufficient condition. The purpose of this note [3] is to give another such condition, related to the definition of binary G-operations of Marczewski [4].

Le φ be a one-one transformation of $\boldsymbol{M}$ into itself. Define φ to be *simple* if $\varphi(a)=b$, $\varphi(b)=a$ for some pair of points $a,b\in M$, and $\varphi(c)=c$ for all other points $c\in M$; that is, φ simply inter-

[1]) It follows by the known theorems on groups every element of which has order at most 2 (cf e. g. L. Pontriagin, *Topological Groups*, Princeton 1940, p. 19, Example 9) that every group of this type is isomorphic to the group of all finite subsets of a set (with symmetric difference as the group operation).

[2]) This fascicle, pp. 199-202.

[3]) Written at the University of Wrocław while the author held a Sheldon Travelling Fellowship from Harvard University.

[4]) E. Szpilrajn-Marczewski, Annales de la Société Polonaise de Mathématique 17, 1938, p. 123-124. A binary G-operation is invariant under all one-one mappings of M into itself, and so is invariant under simple transformations. For an example of an operation defined in the space of integers invariant under simple transformations which is not a G-operation, define sets A and B to be equivalent just if one can be obtained from the other by a finite number of simple transformations, and set $A\circ B=0$ or M according as A and B are not, or are, equivalent.