Application. Supposons qu'on mesure l'angle a en secondes. Pour quelles valeurs entières de a obtiendra-t-on les valeurs de $\cos a$ exprimables par des radicaux réels?

L'angle 2π étant égal à $2^7\cdot 3^4\cdot 5^3$ secondes, on voit que a doit être un multiple de $2^7\cdot 3^4\cdot 5^3/n_0$ où n_0 est le plus grand des diviseurs de $2^7\cdot 3^4\cdot 5^3$ pour lesquels $\varphi(n_0)$ est une puissance de 2. Évidemment $n_0=2^7\cdot 3\cdot 5$, donc les a cherchés sont des multiples de 675″, c'est-à-dire de 11′15″.

A PROOF THAT e^m IS IRRATIONAL

BY

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Niven has given a simple proof that π is irrational[1]). Using the same method[2]) we give here a similarly simple proof that e^m is irrational for any positive integer m[3]).

We define the polynomials:

$$f(x)=\frac{(m^2-x^2)^n}{n!},$$

$$F(x)=f(x)-f'(x)+f''(x)-\ldots+(-1)^n f^{(2n)}(x),$$

where n is a positive integer. Then $f(m)=f'(m)=\ldots=f^{(n-1)}(m)=0$, and $f^{(n)}(x), f^{(n+1)}(x),\ldots,f^{(2n)}(x)$ have integral values for $x=m$. Since $f(x)=f(-x)$ and $f^{(j)}(x)=(-1)^j f(-x)$, we have $f(-m)=f'(-m)=\ldots=f^{(n-1)}(-m)=0$, and $f^{(n)}(x), f^{(n+1)}(x),\ldots,f^{(2n)}(x)$ assume integral values for $x=-m$. Thus, $F(m)$ and $F(-m)$ are *integers*. By elementary calculus we have:

$$\frac{d}{dx}[e^xF(x)]=e^x[F(x)+F'(x)]=e^xf(x).$$

Suppose that $e^m=\frac{a_m}{b_m}$ (for some positive integer m), where a_m and b_m are positive integers.

We have:

$$(*)\qquad a_m b_m\int_{-m}^{+m} e^x f(x)\,dx=a_m b_m[e^xF(x)]_{-m}^{+m}=a_m^2F(m)-b_m^2F(-m).$$

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1) Ivan Niven, *A simple proof that π is irrational*, Bulletin of the American Mathematical Society 53 (1947), p. 509.

2) This method is analogous to that used by Hurwitz for the proof that e is transcendental. Cf. E. Goursat, *Cours d'Analyse Mathématique*, I, 3-me édition, Paris, 1917 p. 210.

3) G. H. Hardy and E. M. Wright write in *An Introduction to the Theory of Numbers* (Oxford 1938, p. 47): „There are other special proofs of the irrationality of e, e^2 and e^3, but it is not much easier to prove e^m irrational, for arbitrary integral m, than it is to prove the full theorem of 11.13". This theorem says: „e is transcendental".

Now $a_m^2 F(m) - b_m^2 F(-m)$ is an *integer*. But for $-m < x < m$:

$$0 < a_m b_m e^x f(x) < \frac{a_m^2 m^{2n}}{n!},$$

so that the integral in (*) is *positive, but arbitrarily small for n sufficiently large.* Thus (*) is false, hence e^m (for $m = 1, 2, 3, \ldots$) is irrational.

CONCERNING THE SYMMETRIC DIFFERENCE IN THE THEORY OF SETS AND IN BOOLEAN ALGEBRAS

BY

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The set $(A-B)+(B-A)=A+B-AB$, i. e. the set of all points which belong to one and only one of the sets A and B, is called the symmetric difference of A and B and will denote in this note by $A \dot{-} B$. The classical books on the General Theory of Sets by Hausdorff, Sierpiński etc. did not treat the symmetric difference [1]), but this operation has proved useful in a number of recent papers. It has been applied especially to two fields: 1° in *measure theory*, the distance of two sets can be defined as the measure of their symmetric difference (Nikodym and Aronszajn) [2]), 2° the class of all subsets of a set (or, more generally, any Boolean algebra) forms a ring in the *algebraic* sense under the operations symmetric difference and multiplication [3]); in particular, symmetric difference is a group operation.

Some investigations in the measure theory suggest a partial analogy between the symmetric difference of sets and the absolute value of the difference of numbers. This idea leads (§ 1) to a new formulation of Kantorovitch-Livenson definition of quasi-analytical operations [4]).

In § 2 I show that the symmetric difference is the only group operation among the binary quasi-analytical operations.

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[1]) Hausdorff treats the symmetric difference only in the last edition of his book (F. Hausdorff, *Mengenlehre*, Berlin-Leipzig 1935, Ergänzungen, p. 276-278).

[2]) See e. g. O. Nikodym, *Sur une généralisation des intégrales de M. J. Radon*, Fundamenta Mathematicae 15 (1930), p. 131-179, especially p. 137, Definition 4.

[3]) See e. g. G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications 25, New York 1940, p. 96.

[4]) L. Kantorovitch and E. Livenson, *Memoir on the Analytical Operations and Projective Sets I*, Fundamenta Mathematicae 18 (1932), p. 214-279, especially p. 239, Definition 1'.