et que τ_i soit un point de densité 1 d'un ensemble dans lequel la fonction $\varphi_{k_i}(t)$ est continue. On a alors pour tout $x \epsilon X_0$

$$U_i(x) = \int_a^b x(t)\varphi_{k_i}(t)dt\,|\varphi_{k_i}(\tau_i)| \to 0;$$

en posant $x_i = \operatorname{sign}\varphi_{k_i}(t)\frac{1}{\sqrt{i}}$ et en appliquant le théorème A, on obtient

$$U_i(x_i) = \int_a^b |\varphi_{k_i}(t)|\,dt\,\frac{|\varphi_{k_i}(\tau_i)|}{\sqrt{i}} \to 0.$$

On a d'autre part $U(x_i) \geqslant \sqrt{i} \to +\infty$, ce qui est impossible.

TWO NOTES ON THE SUMMABILITY OF INFINITE SERIES

BY

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I. ON A THEOREM OF HARDY

1. Hardy's theorem

A series $u_0+u_1+\ldots+u_n+\ldots$ *summable* $(C,1)$ *and with terms* $O(1/n)$ *is convergent*

was historically the first O-Tauberian theorem. Though later more general results were found (like Littlewood's), Hardy's theorem is still useful because of the elementary character of its proof and its sufficiency for many problems. For this reason any simplification of its proof is of interest, if only for didactic purposes, though of course there is no room for basic changes here.

In this note a proof of Hardy's theorem is reproduced which I usually give in my courses.

Let s_n and

$$\sigma_n = (s_0+s_1+\ldots+s_n)/(n+1)$$

be the partial sums and the first arithmetic means of the series $u_0+u_1+\ldots$ We shall also consider the *delayed* arithmetic means

$$(1)\qquad \sigma_{n,k} = \frac{s_n+s_{n+1}+\ldots+s_{n+k-1}}{k} = \frac{(n+k)\sigma_{n+k-1}-n\sigma_{n-1}}{k} = \left(1+\frac{n}{k}\right)\sigma_{n+k-1}-\frac{n}{k}\sigma_{n-1}$$

of the sequence $\{s_n\}$. If k tends to infinity with n in such a way that the ratio n/k remains bounded, then $\sigma_{n,k}$ defines a method of summability which is at least as strong as the method $(C,1)$. For if $\sigma_n \to s$, then the last term in (1) is $s+o(1)$.

The peculiarity of the method (which seems to have been first considered by de la Vallée Poussin [2] for different purposes) is that $\sigma_{n,k}$ is obtained from s_n by adding to it a linear combination with coefficients positive and less than 1 of the terms $u_{n+1}, u_{n+2}, \dots, u_{n+k-1}$. For, as easily seen from (1),

(2) $$\sigma_{n,k} = s_n + \sum_{j=n+1}^{n+k-1}\left(1-\frac{j-n}{k}\right)u_j.$$

2. Let us revert to Hardy's theorem, and let us suppose that $\sigma_n \to s$, $|u_j| \leqslant A/j$ for $j=1,2,\dots$ By the remark just made,

(3) $$|\sigma_{n,k}-s_n| \leqslant \sum_{j=n+1}^{n+k-1}|u_j| \leqslant A\sum_{j=n+1}^{n+k-1}\frac{1}{j} \leqslant A\,\frac{k-1}{n}.$$

Let ε be any positive number and let $k=[\varepsilon n]+1$, where $[x]$ denotes the integer part of x. The last expression in (3) is then $\leqslant An\varepsilon/n = A\varepsilon$. Since $n/k \leqslant n/n\varepsilon = 1/\varepsilon$ is bounded, $\sigma_{n,k}\to s$, so that $\overline{\lim}\,|s-s_n| \leqslant A\varepsilon$. Since ε is arbitrary, $\lim s_n = s$.

3. A similar argument gives Landau's extension of the Hardy's theorem. Landau replaces the condition $|u_j| \leqslant A/j$ by $u_j \leqslant A/j$. An argument parallel to (3) gives $\sigma_{n,k}-s_n \leqslant A(k-1)/n$, and choosing k as before, we get $\overline{\lim}\,(s-s_n) \leqslant A\varepsilon$, $\overline{\lim}\,(s-s_n) \leqslant 0$.

To obtain the inequality $\underline{\lim}\,(s-s_n) \geqslant 0$, we observe that, by (2) $\sigma_{n,k}$ is obtained by *subtracting* from s_{n+k-1} a linear combination with coefficients positive and less than 1 of the terms $u_{n+1},\dots,u_{n+k-1}$, so that $\sigma_{n,k}-s_{n+k-1} \geqslant -A(k-1)/n$. Let us replace here n by $n-k+1$, and set $k=[n\varepsilon]$, where $0<\varepsilon<1$. Since the ratio $(n-k+1)/k < (n+1)/k$ is bounded, we have $\sigma_{n-k+1,k}\to s$ so that

$$\underline{\lim}\,(s-s_n) \geqslant -A\varepsilon/(1-\varepsilon), \qquad \underline{\lim}\,(s-s_n) \geqslant 0.$$

II. ON THE LIMITS OF INDETERMINATION FOR THE METHOD OF RIEMANN

1. Let $\{a_{mn}\}$ be a matrix satisfying the familiar conditions of Toeplitz (see e. g. Zygmund [3], p. 40), so that if $s_0, s_1, \dots$ is any sequence converging to a finite limit s, the sequence of numbers

$$\sigma_m = a_{m0}s_0 + a_{m1}s_1 + \dots a_{mn}s_n + \dots$$

also converges to s as $m\to\infty$. Let now $\{s_n\}$ be any bounded sequence so that the numbers σ_m exist for every m, and let

$$\underline{s} = \underline{\lim}\, s_n, \quad \overline{s} = \overline{\lim}\, s_n, \quad \underline{\sigma} = \underline{\lim}\,\sigma_m, \quad \overline{\sigma} = \overline{\lim}\,\sigma_m.$$

If we set

(1) $$\lambda = \overline{\lim_{m\to\infty}}\,(|a_{m0}|+|a_{m1}|+\dots+|a_{mn}|+\dots)$$

(thus $1 \leqslant \lambda < \infty$) then it is immediate, and very well known, that the interval $(\underline{\sigma}, \overline{\sigma})$ is contained in the interval $(\underline{s}, \overline{s})$ enlarged concentrically λ times. In other words,

(2) $$\frac{1}{2}(\overline{s}+\underline{s}) - \frac{1}{2}\lambda(\overline{s}-\underline{s}) \leqslant \underline{\sigma} \leqslant \overline{\sigma} \leqslant \frac{1}{2}(\overline{s}+\underline{s}) + \frac{1}{2}\lambda(\overline{s}-\underline{s}).$$

By considering sequences $\{s_n\}$ consisting entirely of ± 1 it is easy to see that the number λ defined by (1) is the *least* number λ such that (2) holds for every bounded $\{s_n\}$.

Let us consider a series $u_0+u_1+\dots+u_n+\dots$ with partial sums $s_0, s_1, \dots, s_n, \dots$ The Riemann summability of this series is defined by considering the limit for $\alpha\to 0$ of the expression

(3) $$u_0 + \sum_{n=1}^{\infty} u_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2 = \sum_{n=0}^{\infty} s_n \left\{\left(\frac{\sin n\alpha}{n\alpha}\right)^2 - \left(\frac{\sin (n+1)\alpha}{(n+1)\alpha}\right)^2\right\}.$$

(The equation (3) is certainly valid if the sequence $\{s_n\}$ is bounded, the only case we are interested in here). Though the problem of the limits of indetermination for Riemann's method has been studied, the exact value of λ for it seems not to have been obtained. It has only been proved that $\lambda \leqslant 1+2/\pi^2$ (see Hobson, [1], pp. 224-225, and the bibliography there given).

Theorem. Let $u_0+u_1+\dots$ be a series with partial sums $s_0, s_1, \dots,$ and let $\underline{s} = \underline{\lim}\, s_n$, $\overline{s} = \overline{\lim}\, s_n$. Then, for $\alpha\to 0$, the limits of indetermination $\underline{\sigma}$ and $\overline{\sigma}$ of

$$u_0 + \sum_{n=1}^{\infty} u_n \left(\frac{\sin n\alpha}{n\alpha}\right)^2$$

satisfy the inequality (2) *with* $\lambda = \frac{1}{2}(e^2-5)$. *This is the best possible value of* λ.

2. On account of (3) it is enough to show that

(4) $$\sum_{n=0}^{\infty} \left| \left(\frac{\sin n\alpha}{n\alpha}\right)^2 - \left(\frac{\sin (n+1)\alpha}{(n+1)\alpha}\right)^2 \right| \to \frac{1}{2}(e^2-5)$$

as $\alpha\to 0$. It is easily seen that the left side of (4) tends to the total variation of the function $f(x) = (\sin x)^2/x^2$ over the interval

$(0,+\infty)$. As seen from the graph of the function f, this total variation equals

$$1+2\sum_{j=1}^{\infty}\left(\frac{\sin\alpha_j}{\alpha_j}\right)^2, \tag{5}$$

where $\alpha_1<\alpha_2<\ldots$ are the positive maxima of the function f. These numbers α_j are the positive roots of the equation

$$\sin z = z\cos z, \tag{6}$$

so that the expression (5) can be written

$$1+2\sum_{j=1}^{\infty}\frac{1}{1+\alpha_j^2}. \tag{7}$$

Writing (6) in the form $\tan z = z$ we immediately see that $\pi j<\alpha_j<\pi(j+1)$ for $j=1,2,\ldots$

Let us take temporarily for granted that all the roots of (6) are real. Hence, in addition to the positive roots $\alpha_1,\alpha_2,\ldots$ we also have the negative roots $-\alpha_1,-\alpha_2,\ldots,$ and the root $\alpha_0=0$. Let K_N denote the circle with center at the origin and radius $N\pi$. Since the function $(1+z^2)^{-1}$ is regular outside K_1, we immediately find

$$2\sum_{j=1}^{N}\frac{1}{1+\alpha_j^2}=\frac{1}{2\pi i}\left(\int_{K_{N+1}}-\int_{K_1}\right)\frac{z\sin z}{\sin z - z\cos z}\,\frac{dz}{1+z^2}=$$

$$=\frac{1}{2\pi i}\left(\int_{K_{N+1}}-\int_{K_1}\right)\frac{z\tan z}{\tan z - z}\,\frac{dz}{1+z^2},$$

integration being in the positive direction. The function $\tan z$ is bounded on the circles K_{N+1}, and the last integrand is $O(|z|^{-2})$ for $|z|$ large. It follows that $\int_{K_{N+1}}\to 0$ as $N\to\infty$. Hence the sum (7) is

$$1-\frac{1}{2\pi i}\int_{K_1}\frac{z\tan z}{\tan z - z}\,\frac{dz}{1+z^2}=\frac{1}{2}(e^2-5),$$

as a simple computation of the residues at the points $0,\pm i$ shows.

In order to complete the proof it remains to show that the equation (6), which can also be written $\cot z = 1/z$, has only real roots. If it had a complex root, the same would hold for the equation

$$\cot z = \alpha/z \tag{8}$$

where α is a number sufficiently close to 1. It is therefore enough to show that (8) has no complex root for $0<\alpha<1$. This is easily proved by means of the theorem of Rouché.

For let C_n denote the circle with center $n\pi$ and radius so small that $|\cot z|$ majorizes $|\alpha/z|$ on C_n $(n=0,\pm 1,\ldots;$ the condition $0<\alpha<1$ is essential for $n=0)$. Let C'_N denote the circle $|z|=N\pi-\pi/4$. Along this circle $|\cot z|$ stays away from 0, and so majorizes $|\alpha/z|$, at least for N large enough. Thus, in the region limited externally by C'_{N+1} and internally by the C_n for $n=0,\pm 1,\ldots,\pm N$, the equation (8) has as many roots as the function $\cot z$, that is $2N$. But in this region the equation (8) has already $2N$ real roots whose presence is obvious geometrically. Thus there is no complex root there.

R e m a r k. The method used here can clearly be applied to the summability $(R,2k)$ defining the sum of $u_0+u_1+\ldots$ as

$$\lim_{\alpha\to 0}\left[u_0+\sum_{n=1}^{\infty}u_n\left(\frac{\sin n\alpha}{n\alpha}\right)^{2k}\right].$$

We shall not compute the corresponding constants.

REFERENCES

[1] E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. II.

[2] Ch. J. de la Vallée Poussin, *Leçons sur l'approximation*.

[3] A. Zygmund, *Trigonometrical Series*, 1935.