# *ON A GENERALIZATION OF THEOREMS OF BANACH AND CANTOR-BERNSTEIN*

BY

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S. Banach has proved[1]) the following

*Theorem I. If $\varphi$ is a one-one mapping of a set $\mathfrak{X}$ on a subset of a set $\mathfrak{Y}$ and $\psi$ is a one-one mapping of $\mathfrak{Y}$ on a subset of $\mathfrak{X}$, then there exists a decomposition of $\mathfrak{X}$ and $\mathfrak{Y}$*

$$\text{(i)} \qquad \mathfrak{X} = X_1 + X_2, \qquad \mathfrak{Y} = Y_1 + Y_2$$

such that

$$X_1 \cdot X_2 = 0 = Y_1 \cdot Y_2, \qquad \varphi(X_1) = Y_1, \qquad \psi(Y_2) = X_2.$$

From this theorem follows immediately the well known theorem of Cantor-Bernstein:

*Theorem II. If $\mathfrak{X}$ has the same power as a subset of $\mathfrak{Y}$, and $\mathfrak{Y}$ has the same power as a subset of $\mathfrak{X}$, then $\mathfrak{X}$ and $\mathfrak{Y}$ have the same power.*

In this paper I am going to show that Theorems I and II are particular cases of two more general theorems from the theory of $\sigma$-complete Boolean algebras (Theorems 1 and 2).

**Terminology and notation.** A *Boolean algebra* is a non-empty set $\boldsymbol{A}$ with two operations: complementation $A'$ and addition $A+B$ defined for all $A,B \epsilon \boldsymbol{A}$. These operations satisfy Huntington's axioms:

H1. $\quad A+B=B+A$,

H2. $\quad A+(B+C)=(A+B)+C$,

H3. $\quad (A'+B)'+(A'+B')'=A$.

Multiplication is defined by the formula $AB=(A'+B')'$ and substraction by $A-B=A\cdot B'$. If $A+B=B$, we write $A\subset B$.

[1]) S. Banach, *Un théorème sur les transformations biunivoques*, Fundamenta Mathematicae 6 (1924), pp. 97-101.



Two Boolean algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are *isomorphic* if there exists a one-one mapping $f$ of $\boldsymbol{A}$ on $\boldsymbol{B}$ such that

$$f(A_1)\subset f(A_2) \quad \text{if and only if} \quad A_1\subset A_2.$$

The mapping $f$ is called an *isomorphism* of $\boldsymbol{A}$ on $\boldsymbol{B}$.

A Boolean algebra $\boldsymbol{A}$ is called *$\sigma$-complete* if for every sequence $\{A_n\}$ of elements of $\boldsymbol{A}$ there exists an element $A_0 \epsilon \boldsymbol{A}$ such that

1° $A_n\subset A_0$ for $n=1,2,3,\ldots$,

2° if $A_n\subset A$ for $n=1,2,3,\ldots$, then $A_0\subset A$.

The element $A_0$ will be denoted by $A_1+A_2+\ldots$

For instance, the class of all subsets of a set $\mathfrak{X}$ is a $\sigma$-complete Boolean algebra. This algebra will be denoted by $\boldsymbol{S}(\mathfrak{X})$.

A mapping $f$ of a $\sigma$-complete Boolean algebra $\boldsymbol{A}$ in a $\sigma$-complete Boolean algebra $\boldsymbol{B}$ is called a *$\sigma$-homomorphism* of $\boldsymbol{A}$ in $\boldsymbol{B}$ if

1° $f(A')=(f(A))'$ for each $A\epsilon\boldsymbol{A}$;

2° $f(A_1+A_2+\ldots)=f(A_1)+f(A_2)+\ldots$ for every sequence $\{A_n\}$ of elements of $\boldsymbol{A}$.

Every isomorphism between two $\sigma$-complete Boolean algebras is a $\sigma$-homomorphism.

**1. Lemmas.** Let $\boldsymbol{A}$ be a Boolean algebra and let $E\epsilon\boldsymbol{A}$. The symbol $E\boldsymbol{A}$ will denote the set of all elements $A\epsilon\boldsymbol{A}$ such that $A\subset E$. We define in $E\boldsymbol{A}$ a new operation, complementation $A'_E$, by putting $A'_E=EA'$. Obviously, if $A\epsilon E\boldsymbol{A}$ and $B\epsilon E\boldsymbol{A}$ then $A+B\epsilon E\boldsymbol{A}$ and $A'_E\epsilon E\boldsymbol{A}$. The operations $A'_E$ and $A+B$ on elements of $E\boldsymbol{A}$ satisfy the axioms H1 and H2. They also satisfy H3 since for $A,B\epsilon E\boldsymbol{A}$:

$$(A'_E+B)'_E+(A'_E+B'_E)'_E=E(EA'+B)'+E(EA'+EB')'=$$
$$=E\{[E(A'+B)]'+[E(A'+B')]'\}=E[(A'+B)'+E'+(A'+B')'+E']=$$
$$=EA=A.$$

The set $E\boldsymbol{A}$ is therefore a Boolean algebra. In the case where $\boldsymbol{A}=\boldsymbol{S}(\mathfrak{X})$ and $X\subset\mathfrak{X}$ we have $X\boldsymbol{S}(\mathfrak{X})=\boldsymbol{S}(X)$. If $\boldsymbol{A}$ is $\sigma$-complete, then $E\boldsymbol{A}$ is also $\sigma$-complete.

The following lemmas are obvious:

(A) *Let* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *be* $\sigma$*-complete Boolean algebras,* $B \epsilon \boldsymbol{B}$ *and* $C \epsilon \boldsymbol{C}$. *If* $f$ *and* $g$ *are* $\sigma$*-homomorphisms of* $\boldsymbol{A}$ *in* $B\boldsymbol{B}$ *and of* $B$ *in* $C\boldsymbol{C}$ *respectively, then the mapping* $gf$ *is a* $\sigma$*-homomorphism of* $\boldsymbol{A}$ *in* $g(B)\boldsymbol{C}$.

(B) *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be two Boolean algebras,* $C \epsilon \boldsymbol{A}$ *and* $D \epsilon \boldsymbol{B}$. *If* $C\boldsymbol{A}$ *is an isomorph of* $D'\boldsymbol{B}$, *and* $D\boldsymbol{B}$ *is one of* $C'\boldsymbol{A}$, *then* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are isomorphic.*

Namely, if $f$ is an isomorphism of $C\boldsymbol{A}$ on $D'\boldsymbol{B}$ and $g$ is one of $C'\boldsymbol{A}$ on $D\boldsymbol{B}$, then $h(A)=f(AC)+g(AC')$ (for $A \epsilon \boldsymbol{A}$) is an isomorphism of $\boldsymbol{A}$ on $\boldsymbol{B}$.

(C) *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are two Boolean algebras, and if* $f$ *is an isomorphism of* $\boldsymbol{A}$ *on* $\boldsymbol{B}$, *then* $A\boldsymbol{A}$ *is isomorphic to* $f(A)\boldsymbol{B}$ *for any* $A \epsilon \boldsymbol{A}$.

In fact, the mapping $f(E)$ restricted to $E \subset A$ is an isomorphism of $A\boldsymbol{A}$ on $f(A)\boldsymbol{B}$.

**2. A generalisation of Banach's theorem.** The notions introduced above and Lemma (A) yield the following theorem by suitable modifications of the proof of Banach's Theorem I.

Theorem 1. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be two* $\sigma$*-complete Boolean algebras,* $A \epsilon \boldsymbol{A}$ *and* $B \epsilon \boldsymbol{B}$, *and let* $f$ *and* $g$ *be two* $\sigma$*-homomorphisms of* $\boldsymbol{A}$ *in* $B\boldsymbol{B}$, *and of* $B$ *in* $A\boldsymbol{A}$ *respectively. Then there exist two elements* $C \epsilon \boldsymbol{A}$ *and* $D \epsilon \boldsymbol{B}$ *such that*

$$f(C)=D' \quad \text{and} \quad g(D)=C'.$$

Let $A_1=gf(A')$ and by induction $A_{n+1}=gf(A_n)$ $(n=1,2,3,\ldots)$, and let

$$C=A'+A_1+A_2+A_3+\ldots$$

and $D=(f(C))'$. Obviously $A_n \subset A$, $C \epsilon \boldsymbol{A}$, $D \epsilon \boldsymbol{B}$ and

$$f(C)=D'.$$

On account of (A), $gf$ is a $\sigma$-homomorphism; therefore

$$gf(C)=gf(A')+gf(A_1)+gf(A_2)+\ldots=A_1+A_2+A_3+\ldots=AC$$

and consequently

$$g(D)=g[(f(C))']=A-gf(C)=A-AC=AC'=C',$$

since $C' \subset A$ by definition of $C$. Theorem 1 is proved.

The following theorem on mappings is a consequence of Theorem 1:



Theorem I' [2]). *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be two arbitrary sets,* $X_0 \subset \mathfrak{X}$, $Y_0 \subset \mathfrak{Y}$, *and let* $\varphi$ *and* $\psi$ *be two mappings of* $Y_0$ *into* $\mathfrak{X}$ *and of* $X_0$ *into* $\mathfrak{Y}$ *respectively. Then there exists a decomposition* (i) *such that*

(ii) $\quad X_1 \cdot X_2 = 0 = Y_1 \cdot Y_2, \quad \varphi^{-1}(Y_2)=X_2 \quad \text{and} \quad \varphi^{-1}(X_1)=Y_1.$

The formulas

$$f(X)=\varphi^{-1}(X) \quad \text{for } X \subset \mathfrak{X}, \qquad g(Y)=\psi^{-1}(Y) \quad \text{for } Y \subset \mathfrak{Y}$$

define $\sigma$-homomorphisms of $\boldsymbol{S}(\mathfrak{X})$ in $\boldsymbol{S}(Y_0)=Y_0\boldsymbol{S}(\mathfrak{Y})$ and of $\boldsymbol{S}(\mathfrak{Y})$ in $\boldsymbol{S}(X_0)=X_0\boldsymbol{S}(\mathfrak{X})$ respectively. By Theorem 1, there exist two sets, $X_1 \subset \mathfrak{X}$ and $Y_2 \subset \mathfrak{Y}$, such that

$$f(X_1)=\mathfrak{Y}-Y_2, \qquad g(Y_2)=\mathfrak{X}-X_1.$$

The sets $X_1$, $X_2=\mathfrak{X}-X_1$, $Y_1=\mathfrak{Y}-Y_2$ and $Y_2$ satisfy conditions (ii), q. e. d.

Obviously, Banach's Theorem I is a special case of Theorem I'.

The following example shows that the hypothesis of $\sigma$-completeness of $\boldsymbol{A}$ and $\boldsymbol{B}$ is essential for Theorem I.

Let $N$ be the set of all natural numbers and $A=B=N-(1)$. Further, let $\boldsymbol{A}$ and $\boldsymbol{B}$ denote the same field of all finite subsets of $N$ and their complements. Putting $\varphi(x)=x+1$, and $g(X)=f(X)=\varphi(X)$ for $X \epsilon \boldsymbol{A}=\boldsymbol{B}$, we obtain an isomorphism $f$ of $\boldsymbol{A}$ on $B\boldsymbol{B}$ and an isomorphism $g$ of $\boldsymbol{B}$ on $A\boldsymbol{A}$. It is easy to verify that there exist no sets $C \epsilon \boldsymbol{A}$ and $D \epsilon \boldsymbol{B}$ such that $f(C)=D'$ and $g(D)=C'$.

**3. A generalization of the theorem of Cantor-Bernstein.** Theorem II can be generalized in the following way:

Theorem 2 [3]). *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be* $\sigma$*-complete Boolean algebras,* $A \epsilon \boldsymbol{A}$ *and* $B \epsilon \boldsymbol{B}$. *If* $\boldsymbol{A}$ *is isomorphic to* $B\boldsymbol{B}$ *and* $\boldsymbol{B}$ *to* $A\boldsymbol{A}$, *then* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are isomorphic.*

[2]) This theorem is known. See A. Tarski, *Quelques théorèmes généraux sur les images d'ensembles*, Annales de la Société Polonaise de Mathématique 6, (1928), p. 132-133, and B. Knaster, *Un théorème sur les fonctions d'ensembles* ibidem, p. 133-134.

[3]) This theorem was presented by me at a session of the Warsaw Section of the Polish Mathematical Society on October 4, 1946; cf. *Comptes rendus des Séances des Sections*, Annales de la Société Polonaise de Mathématique 20 (1948), to appear.

Suppose that $f$ and $g$ are isomorphisms of $\boldsymbol{A}$ on $B\boldsymbol{B}$ and of $\boldsymbol{B}$ on $A\boldsymbol{A}$ respectively. Since $f$ and $g$ are $\sigma$-homomorphisms, then there exist by Theorem 1 two elements $C\epsilon\boldsymbol{A}$ and $D\epsilon\boldsymbol{B}$ such that

$$f(C)=D' \quad \text{and} \quad g(D)=C'.$$

By Lemma (C), $C\boldsymbol{A}$ is isomorphic to $D'\boldsymbol{B}$ and $D\boldsymbol{B}$ is isomorphic to $C'\boldsymbol{A}$. By Lemma (B), the algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are isomorphic, q. e. d.

In order to prove that the theorem of Cantor-Bernstein is a particular case of Theorem 2, it is enough to note that two arbitrary sets $X$ and $Y$ have the same power if and only if $\boldsymbol{S}(X)$ and $\boldsymbol{S}(Y)$ are isomorphic [4].

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two abstract sets, $X\subset\mathfrak{X}$ and $Y\subset\mathfrak{Y}$. Suppose that $\overline{\overline{\mathfrak{X}}}=\overline{\overline{Y}}$ and $\overline{\overline{\mathfrak{Y}}}=\overline{\overline{X}}$, i.e. that $\boldsymbol{S}(\mathfrak{X})$ is isomorphic to $\boldsymbol{S}(Y)=Y\boldsymbol{S}(\mathfrak{Y})$, and $\boldsymbol{S}(\mathfrak{Y})$ is isomorphic to $\boldsymbol{S}(X)=X\boldsymbol{S}(\mathfrak{X})$. By Theorem 2, $\boldsymbol{S}(\mathfrak{X})$ is isomorphic to $\boldsymbol{S}(\mathfrak{Y})$, i.e. $\overline{\overline{\mathfrak{X}}}=\overline{\overline{\mathfrak{Y}}}$, q. e. d.

**4. An unsolved problem.** The question arises whether other theorems from the general theory of sets which can be expressed in terms of the theory of Boolean algebras may be generalized in the same way as the theorem of Cantor-Bernstein. For instance, the well known theorem of Bernstein on cardinal numbers:

$$2\mathfrak{m}=2\mathfrak{n} \quad \textit{implies} \quad \mathfrak{m}=\mathfrak{n}$$

can be expressed as follows:

(*) *If $A$ and $B$ are two elements of a Boolean algebra $\boldsymbol{A}$ such that $A\boldsymbol{A}$ is an isomorph of $A'\boldsymbol{A}$ and $B\boldsymbol{A}$ is an isomorph of $B'\boldsymbol{A}$, then $A\boldsymbol{A}$ and $B\boldsymbol{A}$ are isomorphic.*

This theorem may be deduced from Bernstein's theorem in case where $\boldsymbol{A}$ is totally additive (i. e. complete) and atomic. The following question is unsolved:

**P25.** Whether the theorem (*) is true for all infinitely or totally additive Boolean algebras?

[4]) E. Szpilrajn-Marczewski, *On the isomorphism and the equivalence of classes and sequences of sets,* Fundamenta Mathematicae 32 (1939), pp. 133-148, in particular p. 137, (i).

# UN THÉORÈME SUR LA CONVERGENCE UNIFORME DANS L'INTÉRIEUR

PAR

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On dit que la suite de fonctions (réelles ou complexes) définies sur un ensemble ouvert $G$ situé dans un espace euclidien à un nombre quelconque de dimensions *converge uniformément dans l'intérieur* de $G$ si elle converge uniformément sur tout sous-ensemble fermé et borné de $G$. Cette dénomination a été introduite par M. Montel [1]) et elle est devenue courante dans la théorie des fonctions analytiques.

Plusieurs auteurs ont traité certaines classes de fonctions (entières, holomorphes dans le cercle-unité, etc.) comme des espaces métriques, de façon que la convergence dans ces espaces coïncidait avec celle uniforme dans l'intérieur [2]). Il est facile de constater que les espaces définis par eux sont des espaces du type $(F)$ au sens de Banach [3]).

La question s'impose, si la convergence uniforme dans l'intérieur ne se prête à une caractérisation au moyen de la norme du type $(B)$, c'est-à-dire de la norme de Banach. Rappelons que, dans un espace $E$ dit linéaire ou vectoriel [4]), on entend par *norme du type* $(B)$ toute fonctionnelle réelle $[x]$, définie pour tout $x\epsilon E$ et assujettie aux conditions [5]):

(a) $[x_1+x_2]\leqslant[x_1]+[x_2]$,

(b) $[\lambda\cdot x]=|\lambda|\cdot[x]$ pour tout nombre $\lambda$,

(c) $[x]=0$ équivaut à $x=0$.

[1]) P. Montel, *Leçons sur les familles normales de fonctions analytiques*, Paris 1927, p. 26.

[2]) M. Fréchet, *Espaces abstraits*, Paris 1928, p. 87; S. Kierst et E. Szpilrajn-Marczewski, Fundamenta Mathematicae 21 (1933), p. 281; cf. aussi S. Saks et A. Zygmund, *Funkcje analityczne*, Monografie Matematyczne 10, Warszawa 1938, p. 49, ćwiczenie 3, et p. 116, ćwiczenie 3 (en polonais).

[3]) S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne 1, Warszawa 1932, p. 35.

[4]) S. Banach, op. cit., p. 26.

[5]) S. Banach, op. cit., p. 53.