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MEASURES IN NON-SEPARABLE METRIC SPACES

BY

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In this paper we call a *measure* every σ-additive set function $\mu(X)$, such that $0 \leqslant \mu(X) \leqslant +\infty$, defined on a σ-additive field of subsets of a set $\mathfrak{X}$. A measure is said to be *σ-finite*, if $\mathfrak{X}$ is the sum of an enumerable sequence of sets of finite measure.

A measure on the field of all Borel subsets of a metric space $\mathfrak{X}$ is called a *Borel measure* in $\mathfrak{X}$.

The chief problem of this paper [1]) is a decomposition of any metric space $\mathfrak{X}$ with a σ-finite Borel measure μ:

(1) $\mathfrak{X} = N + S$, where $\mu(N) = 0$ and S is separable.

We shall prove that, roughly speaking, this problem is equivalent to the known *generalized problem of measure* of Banach [2]) (see Theorems III, IV and V). In particular the decomposition (1) is possible for every metric space $\mathfrak{X}$ for which the answer to Banach's problem is negative, e. g. for each $\mathfrak{X}$ of power $\aleph_1$.

Therefore, the results of this paper reduce, in practice, the examining of Borel measures in metric spaces to the separable case.

Two ideas play an essential part in our proofs: a certain method of Banach concerning measures in abstract sets and a theorem of Montgomery on non-separable metric spaces (see p. 135).

We wish to thank Professor B. Knaster, whose questions and remarks have contributed to the solution of the chief problem of this paper.

1. Lemmas on σ-finite measures. We shall establish two simple lemmas.

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[1]) Presented to the Polish Mathematical Society, Wrocław Section, on October 30, 1947, and to the Warsaw Section on December 5, 1947.

[2]) See Banach et Kuratowski [2], Banach [1], Ulam [11], Marczewski [4], p. 308, Marczewski et Sierpiński [6], Sierpiński [9], p. 107.

Numbers in brackets refer to the bibliography at the end of the paper.

1. *Every σ-finite measure μ in a field $\boldsymbol{K}$ may be expressed as a series of finite measures μ_n in $\boldsymbol{K}$:*

$$\mu(X)=\mu_1(X)+\mu_2(X)+\dots \qquad \textit{for each} \qquad X\in\boldsymbol{K}.$$

In fact, μ being σ-finite, there is a sequence of sets $X_n\in\boldsymbol{K}$ such that $\mu(X_n)<\infty$. Putting $\mu_n(X)=\mu(XX_n)$, we obtain the required decomposition.

We say that a Borel measure μ in $\mathfrak{X}$ is *everywhere positive* in a Borel set $X\subset\mathfrak{X}$, if $\mu(U)>0$ for every non-void subset U of X which is open in X.

2. *If a σ-finite Borel measure μ in a metric space $\mathfrak{X}$ is everywhere positive in X, then X is a separable set.*

For any finite Borel measure in $\mathfrak{X}$, each class of disjoint sets of positive measure is at most enumerable. On account of Lemma 1, the same holds for any σ-finite Borel measure. Then, from hypothesis it follows that every class of disjoint sets, which are contained and open in X, is at most enumerable (or, in other terms, that X possesses the so-called Souslin property [3])). X, as a metric space, is therefore separable.

2. Addition theorem. We say that a cardinal number $\mathfrak{m}$ *has measure zero* if every finite measure μ, defined for all subsets of any set $\mathfrak{Y}$ of power $\mathfrak{m}$ and vanishing for all one-point sets, vanishes identically [4]).

It is easy to see that, if $\mathfrak{m}$ has measure zero and $\mathfrak{n}<\mathfrak{m}$, then $\mathfrak{n}$ has also measure zero.

Ulam has proved that every cardinal number less than the first aleph inaccessible in the weak sense [5]) has measure zero [6]).

[3]) See e. g. Marczewski [5], p. 128 and 130.

[4]) Obviously, if a set $\mathfrak{Y}$ fulfils this condition, then every set of the same power fulfils it also.

[5]) $\mathfrak{p}=\aleph_\lambda>\aleph_0$ is called *inaccessible in the weak sense*, if λ is a limit number and if the condition $\mathfrak{p}_t<\mathfrak{p}$, where t runs over a set T of a power less than $\mathfrak{p}$, implies that $\sum_{t\in T}\mathfrak{p}_t<\mathfrak{p}$. See Tarski [10], p. 69.

[6]) See Ulam [11], p. 141. Satz (A). We use the term „cardinal of measure zero" instead of Ulam's term „non-measurable cardinal". Our term seems to be more adequate: compare e. g. the Ulam's Lemma 1 (ibidem, p. 144) which receives now the following intuitive form: *Let $\mathfrak{m}$ be a cardinal number of measure zero. If a cardinal $\mathfrak{n}$ is the sum of $\mathfrak{m}$ cardinals of measure zero, then $\mathfrak{n}$ is also of measure zero.* See likewise our Theorem I.

In particular $\aleph_1$ and $\aleph_\omega$ have measure zero. The analogous question for the power of the continuum [7]) belongs to-day to the classical problems of the General Theory of Sets.

Theorem I. Let μ be a σ-finite Borel measure in a metric space $\mathfrak{X}$, and let $\boldsymbol{G}$ be a class of open subsets of $\mathfrak{X}$ of measure μ zero. If the power of $\boldsymbol{G}$ has measure zero, then the sum N of all $G\in\boldsymbol{G}$ is of measure μ zero.

Proof. By Lemma 1 it can be assumed without any loss of generality that μ is finite. Let G_ξ $(0\leqslant\xi<\gamma)$ be a transfinite sequence of all $G\in\boldsymbol{G}$ and let

$$H_\eta=G_\eta-\sum_{\xi<\eta}G_\xi \quad \text{for} \quad 0\leqslant\eta<\gamma.$$

Obviously

$$N=\sum_{\eta<\gamma}H_\eta$$

and $\mu(H_\eta)=0$, since $H_\eta\subset G_\eta$. Let $\mathfrak{Y}$ denote the set of all ordinals $\eta<\gamma$. If $Y\subset\mathfrak{Y}$, then, by a theorem of Montgomery [8]), the set

$$H(Y)=\sum_{\eta\in Y}H_\eta$$

is an F_σ.

Putting $\nu(Y)=\mu[H(Y)]$ for any $Y\subset\mathfrak{Y}$ [9]) we obtain a finite measure ν defined for all subsets of $\mathfrak{Y}$. Moreover, $\nu[(\eta)]=\mu(H_\eta)=0$ for every $\eta\in\mathfrak{Y}$. Hence $\mu(N)=\nu(\mathfrak{Y})=0$, since the power of $\mathfrak{Y}$ has measure zero.

3. Separability character. If $\boldsymbol{G}$ is a class of sets, we denote by $\sum(\boldsymbol{G})$ the sum of all sets $G\in\boldsymbol{G}$.

A class $\boldsymbol{B}$ of open subsets of a metric space $\mathfrak{X}$ is called a *basis of* $\mathfrak{X}$, if for every open $G\subset\mathfrak{X}$ there exists a subclass $\boldsymbol{G}$ of $\boldsymbol{B}$ such that $G=\sum(\boldsymbol{G})$.

[7]) See Ulam [11], p. 141 (II).

[8]) Montgomery [8], Lemma 2, p. 528. Montgomery's hypothesis on the increase of the sequence in question is superfluous. Cf. Kuratowski [3], p. 534 et 537, 1°.

[9]) This method is originally due to Banach. Cf. Banach [1], p. 101, and Ulam [11], p. 144, footnote 2.

Theorem II. The following four properties[10]) *are equivalent for metric spaces:*

(δ) *there exists a dense subset of $\mathfrak{X}$, whose power has measure zero.*

(β) *there exists a basis* **B**, *whose power has measure zero.*

(λ) *for each class* **G** *of open sets there exists a subclass* $\boldsymbol{H}\subset\boldsymbol{G}$ *such that* $\sum(\boldsymbol{G})=\sum(\boldsymbol{H})$ *and whose power has measure zero.*

(σ) *the power of any class of disjoint open sets has measure zero*[11]).

Proof. We shall prove the following implications:

$$(\delta)\rightarrow(\beta)\rightarrow(\lambda)\rightarrow(\sigma)\rightarrow(\delta).$$

The proofs of the implications $(\delta)\rightarrow(\beta)\rightarrow(\lambda)$ are the same as in the case of spaces separable in the ordinary sense. The implication $(\lambda)\rightarrow(\sigma)$ is obvious.

Let us prove that $(\sigma)\rightarrow(\delta)$. $\mathfrak{X}$ being a metric space with the property (σ), there exists for each positive integer n a set D_n, whose power has measure zero, and such that for each $x\epsilon\mathfrak{X}$ there is a $y\epsilon D_n$ with $\varrho(x,y)<\frac{1}{n}$[12]). The set $D=D_1+D_2+\dots$ is dense in $\mathfrak{X}$ and its power has measure zero[13]).

The smallest power of a basis of $\mathfrak{X}$ may be called the *separability character of* $\mathfrak{X}$. Hence, the property (β) of a space $\mathfrak{X}$ asserts that the separability character of $\mathfrak{X}$ is of measure zero.

In particular, if the power of a space has measure zero, the separability character of $\mathfrak{X}$ is also of measure zero.

4. Decomposition theorems. The answer to the problem of decomposition (1) is given by Theorems III, IV and V.

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[10]) The property (λ) is analogous to the well known theorem of Lindelöf and the property (σ) to the well known property of Souslin.

[11]) Obviously, Theorem II is a particular case of a general theorem concerning the spaces whose separability character is less than a cardinal $\mathfrak{n}$. Namely, we may formulate analogously the properties $(\delta_\mathfrak{n})$, $(\beta_\mathfrak{n})$, $(\lambda_\mathfrak{n})$, $(\sigma_\mathfrak{n})$ and we can prove for any $\mathfrak{n}$ the implications $(\delta_\mathfrak{n})\rightarrow(\beta_\mathfrak{n})\rightarrow(\lambda_\mathfrak{n})\rightarrow(\sigma_\mathfrak{n})$.

Moreover, if the cardinal $\mathfrak{n}$ fulfils the following condition:

(*) $\mathfrak{n}_1+\mathfrak{n}_2+\dots<\mathfrak{n}$ for each sequence of cardinals $\mathfrak{n}_j<\mathfrak{n}$

then also $(\sigma_\mathfrak{n})\rightarrow(\delta_\mathfrak{n})$.

[12]) $\varrho(x,y)$ denotes the distance between x and y.

[13]) See Ulam's lemma cited above, p. 134, footnote 6. Compare also our footnote 11, condition (*).

Theorem III. If the separability character of a metric space $\mathfrak{X}$ has measure zero and if μ is a σ-finite Borel measure in $\mathfrak{X}$, then there exists a decomposition (1).

We shall prove, more precisely, that

(i) *the sum N of all open sets of measure μ zero has also measure μ zero.*

(ii) *the measure μ is everywhere positive in the set $S=\mathfrak{X}-N$ and therefore S is separable* (by Lemma 2).

By Theorem II, the space $\mathfrak{X}$ has the property (λ) and consequently there exists a class **G** of open sets such that the power of **G** is of measure zero,

$$\mu(G)=0 \quad \text{for} \quad G\epsilon\boldsymbol{G}, \quad \text{and} \quad \sum(\boldsymbol{G})=N.$$

By Theorem I, $\mu(N)=0$, which establishes the proposition (i).

Now, let U be a non-void subset of $\mathfrak{X}-N$, open in $\mathfrak{X}-N$; then $U=G-N$, where G is open in $\mathfrak{X}$. By definition of N, and by (i), we have $\mu(G)>0$ and $\mu(GN)=0$. Hence $\mu(U)=\mu(G)>0$, which establishes proposition (ii).

The two following theorems are converses of Theorem III.

Theorem IV. If for every finite Borel measure μ in a metric space $\mathfrak{X}$ there exists a decomposition (1), *then the separability character of $\mathfrak{X}$ has measure zero.*

Proof. Let **G** be an arbitrary class of disjoint open subsets of $\mathfrak{X}$ and let us choose one point from every set belonging to **G**. We denote by I the set of all chosen points. Let us consider any finite measure ν defined for all subsets of I and vanishing for all one-point sets.

The formula $\mu(X)=\nu(XI)$ defines a finite Borel measure μ in $\mathfrak{X}$. By hypothesis, there exists a decomposition $\mathfrak{X}=N+S$, where $\mu(N)=0$, and where S is separable.

The set IS, as an isolated subset of the separable space S, is at most enumerable; therefore $\mu(IS)=\nu(IS)=0$.

Consequently,

$$\nu(I)=\mu(\mathfrak{X})=\mu(N)+\mu(S)=\mu(S-I)+\mu(IS)=\nu[I(S-I)]=\nu(0)=0.$$

On account of the arbitrariness of ν, the power of I has measure zero. Since I and **G** have the same power, the space $\mathfrak{X}$ has the property (σ) and, by Theorem II, the separability character of $\mathfrak{X}$ has measure zero.

Theorem V. If for every space $\mathfrak{X}$ of power $\mathfrak{m}$ and for every finite Borel measure μ in $\mathfrak{X}$ there exists a decomposition (1), *then the cardinal number $\mathfrak{m}$ has measure zero.*

Proof. Let $\mathfrak{X}$ be a set of power $\mathfrak{m}$. Consider $\mathfrak{X}$ as a space with the trivial metric:

$$\varrho(x,x)=0 \quad \text{and} \quad \varrho(x_1,x_2)=1 \quad \text{for} \quad x_1 \neq x_2$$

and let μ be an arbitrary finite measure defined for all subsets of $\mathfrak{X}$ and vanishing for all one-point sets. The measure μ being a Borel measure in $\mathfrak{X}$, there exists a decomposition (1).

Since every separable subset of $\mathfrak{X}$ is at most enumerable, $\mu(\mathfrak{X})=\mu(S)=0$. Thus, the power of $\mathfrak{X}$ has measure zero.

5. Two-valued measures. A measure μ is called *two-valued* if it assumes at most two values: 0 and 1. By definition, a cardinal number $\mathfrak{m}$ *has two-valued measure zero* if every two-valued measure, defined for all subsets of a set of power $\mathfrak{m}$ and vanishing for all one-point sets, vanishes identically.

Obviously, if $\mathfrak{m}$ has measure zero, it has also two-valued measure zero. As Tarski and Ulam have proved, if a cardinal $\mathfrak{m}$ has two-valued measure zero, $2^{\mathfrak{m}}$ has the same property [14]). Every cardinal less than the firts aleph inaccessible in the strict sense [15]) has two-valued measure zero. In particular the power of the continuum has two-valued measure zero.

It is easy to verify that all the definitions, the theorems and the proofs of paragraphs 2, 3 and 4 run the same way, if we replace everywhere the term „measure" by the term „two-valued measure".

Besides, here the consideration of the separability character appears superfluous, since the separability character of a metric space $\mathfrak{X}$ has two-valued measure zero if and only if the power of $\mathfrak{X}$ has the same property. In fact, if $\mathfrak{m}$ is the separability character of an infinite space $\mathfrak{X}$, then the power of $\mathfrak{X}$ is $\overline{\overline{\mathfrak{X}}} \leqslant \mathfrak{m}^{\aleph_0} \leqslant 2^{\mathfrak{m}}$, and therefore, if $\mathfrak{m}$ has two-valued maesure zero, the power of $\mathfrak{X}$ is of two-valued measure too. The converse is evident.

[14]) Ulam [12], p. 146. For the applications of powers of two-valued measure zero, see Mazur [7] and two papers of Sikorski to appear in Fundamenta Mathematicae 35.

[15]) A cardinal number $\mathfrak{p} > \aleph_0$ is called *inaccessible in the strict sense* if it is inaccessible in the weak sense and if, moreower, $\mathfrak{m}^{\mathfrak{n}} < \mathfrak{p}$ for every $\mathfrak{m} < \mathfrak{p}$ and $\mathfrak{n} < \mathfrak{p}$. See Tarski [10], p. 69.

Obviously, if a two-valued Borel measure μ in $\mathfrak{X}$ is everywhere positive on a set E, then E consists of a single point. The „two-valued" Theorem III and the proposition (ii) take, then, the following form:

Theorem VI. If the power of a metric space $\mathfrak{X}$ has two-valued measure zero, then every two-valued Borel measure μ in $\mathfrak{X}$ is trivial, i. e. either $\mu(\mathfrak{X})=0$ or there is a point $x_0 \epsilon \mathfrak{X}$ with $\mu[(x_0)]=1$ (and therefore $\mu[\mathfrak{X}-(x_0)]=0$).

In particular, any two-valued Borel measure in a metric space of the power of the continuum is trivial.

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