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Lyapunov functional for a system with k-non-commensurate neutral time delays*

by

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Abstract: The paper presents a method of determining the Lyapunov quadratic functional for linear time-invariant system with k-non-commensurate neutral type time delays. The Lyapunov functional is constructed for its given time derivative, which is calculated on the trajectory of the system with k-non-commensurate neutral type time delays. The presented method gives analytical formulas for the coefficients of the Lyapunov functional.

Keywords: Lyapunov functional, time delay system, neutral system, LTI system.

1. Introduction

Lyapunov quadratic functionals are used to test the stability of systems, in computation of the critical delay values for time delay systems, in computation of the exponential estimates for the solutions of time delay systems, in calculation of the robustness bounds for uncertain time delay systems, to calculation of a quadratic performance index of quality for the process of parametric optimization for time delay systems. We construct the Lyapunov functionals for the system with time delay with a given time derivative. For the first time such Lyapunov functional was introduced by Repin (1965) for the case of retarded time delay linear systems with one delay. Repin (1965) delivered also the procedure for determination of coefficients of the functional. Duda (1986) used the Lyapunov functional, which was proposed by Repin, for the calculation of the value of a quadratic performance index of quality in the process of parametric optimization for systems with time delay of retarded type and extended the results to the case of neutral type time delay system in Duda (1988) and to the case of linear time invariant system with two lumped retarded type time delays in Duda (2010). In Infante and Castelan (1978), construction of the Lyapunov

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functional is based on a solution of a matrix differential-difference equation on a finite time interval. This solution satisfies symmetry and boundary conditions. Kharitonov and Zhabko (2003) extended the results of Infante and Castelan (1978) and proposed a procedure of construction of quadratic functionals for linear retarded type delay systems which could be used for the robust stability analysis of time delay systems. This functional was expressed by means of Lyapunov matrix, which depended on the fundamental matrix of time delay system. Kharitonov (2005) extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and in Kharitonov (2008) to the neutral type time delay systems with discrete and distributed delay. Kharitonov and Hinrichsen (2004) used the Lyapunov matrix to derive exponential estimates for the solutions of exponentially stable time delay systems. Kharitonov and Plischke (2006) formulated the necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of retarded system with one delay. The numerical scheme for construction of the Lyapunov functionals has been proposed in Gu (1997).This method starts with the discretisation of the Lyapunov functional. The scheme is based on linear matrix inequality (LMI) techniques. Fridman (2001) introduced the Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems with discrete and distributed delays, which were based on equivalent descriptor form of the original system and obtained delay-dependent and delay-independent conditions in terms of LMI. Ivanescu et al. (2003) proceeded with the delay-depended stability analysis for linear neutral systems, constructed the Lyapunov functional and derived sufficient delay-dependent conditions in terms of linear matrix inequalities (LMIs). Han (2004a) obtained a delay-dependent stability criterion for neutral systems with time varying discrete delay. This criterion was expressed in the form of LMI and was obtained using the Lyapunov direct method. Han (2004b) investigated the robust stability of uncertain neutral systems with discrete and distributed delays, which has been based on the descriptor model transformation and the decomposition technique, and formulated the stability criteria in the form of LMIs. Han (2005a) considered the stability for linear neutral systems with norm-bounded uncertainties in all system matrices and derived a new delay-dependent stability criterion. Neither model transformation nor bounding technique for cross terms is involved through derivation of the stability criterion. Han (2005b) developed the discretized Lyapunov functional approach to investigation of the stability of linear neutral systems with mixed neutral and discrete delays. Stability criteria, which are applicable to linear neutral systems with both small and non-small discrete delays are formulated in the form of LMIs. Han (2009a) studied the problem of stability of linear time delay systems, both retarded and neutral types, using the discrete delay N-decomposition approach to derive some new more general discrete delay dependent stability criteria. Han (2009b) employed the delay-decomposition approach to derive some improved stability criteria for linear neutral systems and to deduce some sufficient conditions for the existence of a delayed state feedback controller, which ensure asymptotic stability and a prescribed H∞ performance level of the corresponding closed-loop system. Gu and Liu (2009) investigated the stability of coupled differential-functional equations using the discretized Lyapunov functional method and delivered the stability condition in the form of LMI, suitable for numerical computation.

This paper presents a method of determining the Lyapunov functional for linear dynamic system with k-non-commensurate neutral type time delays. The novelty of the result lies in the extension of the Repin method to the system with k-non-commensurate neutral type time delays. To the best of author's knowledge, such extension has not been reported in the literature. An example illustrating the method is also presented.

2. Formulation of the problem

Let us consider a linear system with k-non-commensurate neutral type time delays, whose dynamics is described by the equation

$$\begin{cases} \frac{dx(t)}{dt} - \sum_{i=1}^{k} B_i \frac{dx_t(-\tau_i)}{dt} = Ax(t) + \sum_{i=1}^{k} A_i x_t(-\tau_i) \\ x(t_0) = x_0 \\ x_{t_0} = \Phi \in W^{1,2}([-\tau_k, 0), \mathbb{R}^n) \end{cases} \tag{1}$$

$$\begin{aligned} &\text{for } t \geq t_0,\ x(t) \in \mathbb{R}^n,\ A,\ A_i,\ B_i \in \mathbb{R}^{n\times n},\ i = 1, ..., k, \\ &0 \leq \tau_1 \leq ... \leq \tau_i \leq ... \leq \tau_k, \\ &x_t \in W^{1,2}([-\tau_k, 0), \mathbb{R}^n), \quad x_t(\theta) = x(t+\theta) \quad \text{for} \quad \theta \in [-\tau_k, 0). \end{aligned}$$

Here, $W^{1,2}([-\tau_k, 0), \mathbb{R}^n)$ is a space of all absolutely continuous functions with derivatives in a space of Lebesgue square integrable functions on interval $[-\tau_k, 0)$ with values in $\mathbb{R}^n$.

We introduce a new variable y, defined by the formula

$$y(t) = x(t) - \sum_{i=1}^{k} B_i x_t(-\tau_i) \quad \text{for} \quad t \geq t_0. \tag{2}$$

Thus, the equations (1) take the form

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + \sum_{i=1}^{k} (A_i + B_i) x_t(-\tau_i) \\ y(t) = x(t) - \sum_{i=1}^{k} B_i x_t(-\tau_i) \\ y(t_0) = x_0 - \sum_{i=1}^{k} B_i \Phi(-\tau_i) \\ x_{t_0} = \Phi. \end{cases} \tag{3}$$

The state of the system (3) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \quad \text{for} \quad t \geq t_0. \tag{4}$$

The state space is defined by the formula

$$X = \mathbb{R}^n \times W^{1,2}([-\tau_k, 0), \mathbb{R}^n). \tag{5}$$

On the state space X we define a Lyapunov functional, positively defined, differentiable, whose derivative, computed on the trajectory of the system (3), is negatively defined

$$V(S(t)) = y^T(t)\alpha y(t) + \int_{-\tau_k}^{0} y^T(t)\beta(\theta)x_t(\theta)d\theta + \tag{6}$$
$$+ \int_{-\tau_k}^{0} x_t^T(\theta)\gamma(\theta)x_t(\theta)d\theta + \int_{-\tau_k}^{0}\int_{\theta}^{0} x_t^T(\theta)\delta(\theta,\sigma)x_t(\sigma)d\sigma d\theta$$

for $t \geq t_0$, where

$$\alpha = \alpha^T \in \mathbb{R}^{n\times n},\ \beta,\ \gamma \in C^1([-\tau_k, 0], \mathbb{R}^{n\times n}),\ \gamma(\theta) = \gamma^T(\theta)$$
$$\delta \in C^1(\Omega, \mathbb{R}^{n\times n}), \quad \Omega = \{(\theta,\sigma) :\ \theta \in [-\tau_k, 0],\ \sigma \in [\theta, 0]\},$$

and C^1 is a space of continuous functions with continuous derivative.

3. Designation of the coefficients of the Lyapunov functional

We compute the derivative of the functional (6) on the trajectory of the system (3) according to the formula

$$\frac{dV(S(t))}{dt} = grad(V(S(t))\frac{dS(t)}{dt} \quad \text{for} \quad t \geq t_0. \tag{7}$$

Derivative of the functional (6), calculated on the basis of the formula (7), is given by the formula

$$\frac{dV(S(t))}{dt} = y^T(t)\left[A^T\alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} + \gamma(0)\right]y(t) +$$
$$+ y^T(t)\left[2\alpha\left(A_k + B_k\right) + \beta(0)B_k + 2\gamma(0)B_k - \beta(-\tau_k)\right]x_t(-\tau_k) +$$
$$+ \sum_{i=1}^{k-1} y^T(t)\left[2\alpha(A_i + B_i) + \beta(0)B_i + 2\gamma(0)B_i\right]x_t(-\tau_i) +$$
$$+ x_t^T(-\tau_k)\left[B_k^T\gamma(0)B_k - \gamma(-\tau_k)\right]x_t(-\tau_k) +$$
$$+ \sum_{i=1}^{k-1} x_t^T(-\tau_k)B_k^T 2\gamma(0)B_i x_t(-\tau_i) + \sum_{i,j=1}^{k-1} x_t^T(-\tau_i)B_i^T\gamma(0)B_j x_t(-\tau_j) +$$

$$+\int_{-\tau_k}^{0} y^T(t)\left[A^T\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+\delta^T(\theta,0)\right]x_t(\theta)d\theta+ \tag{8}$$

$$+\int_{-\tau_k}^{0} x_t^T(-\tau_k)[(A_k+B_k)^T\beta(\theta)+B_k^T\delta^T(\theta,0)-\delta(-\tau_k,\theta)]x_t(\theta)d\theta+$$

$$+\int_{-\tau_k}^{0}\sum_{i=1}^{k-1} x_t^T(-\tau_k)[(A_i+B_i)^T\beta(\theta)+B_i^T\delta^T(\theta,0)]x_t(\theta)d\theta+$$

$$-\int_{-\tau_k}^{0} x_t^T(\theta)\frac{d\gamma(\theta)}{d\theta}x_t(\theta)d\theta-\int_{-\tau_k}^{0}\int_{\theta}^{0} x_t^T(\theta)\left[\frac{\partial\delta(\theta,\sigma)}{\partial\theta}+\frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right]x_t(\sigma)d\sigma d\theta$$

for $t\geq t_0$.

We identify the coefficients of the functional (6) assuming that the derivative (8) satisfies the relationship

$$\frac{dV(S(t))}{dt}=-y^T(t)Wy(t)\qquad for\quad t\geq t_0 \tag{9}$$

where $W\in\mathbb{R}^{n\times n}$ is a symmetric positively defined matrix.

When the system (3) is asymptotically stable and the relationship (9) holds, one can easily determine the value of a square indicator of quality of parametric optimization, knowing the Lyapunov functional (6), because

$$J=\int_{t_0}^{\infty} y^T(t)Wy(t)dt=V(S(t_0)). \tag{10}$$

From equation (8) and (9) we obtain the system of equations

$$A^T\alpha+\alpha A+\frac{\beta(0)+\beta^T(0)}{2}+\gamma(0)=-W \tag{11}$$

$$2\alpha\,(A_i+B_i)+\beta(0)B_i+2\gamma(0)B_i=0 \text{ for } i=1,...,k-1 \tag{12}$$

$$2\alpha\,(A_k+B_k)+\beta(0)B_k+2\gamma(0)B_k-\beta(-\tau_k)=0 \tag{13}$$

$$B_k^T\gamma(0)B_k-\gamma(-\tau_k)=0 \tag{14}$$

$$B_k^T 2\gamma(0)B_i=0\quad\text{for } i=1,...,k-1 \tag{15}$$

$$B_i^T\gamma(0)B_j=0\quad\text{for } i,j=1,...,k-1 \tag{16}$$

$$\frac{d\gamma(\theta)}{d\theta}=0 \tag{17}$$

$$A^T\beta(\theta)-\frac{d\beta(\theta)}{d\theta}+\delta^T(\theta,0)=0 \tag{18}$$

$$(A_k + B_k)^T \beta(\theta) + B_k^T \delta^T(\theta, 0) - \delta(-\tau_k, \theta) = 0 \tag{19}$$

$$(A_i + B_i)^T \beta(\theta) + B_i^T \delta^T(\theta, 0) = 0 \quad \text{for } i = 1, ..., k-1 \tag{20}$$

$$\frac{\partial \delta(\theta, \sigma)}{\partial \theta} + \frac{\partial \delta(\theta, \sigma)}{\partial \sigma} = 0 \tag{21}$$

for $\theta \in [-\tau_k, 0]$, $\sigma \in [\theta, 0]$.

Equation (12) implies that

$$2\alpha \sum_{i=1}^{k-1} (A_i + B_i) + \beta(0) \sum_{i=1}^{k-1} B_i + 2\gamma(0) \sum_{i=1}^{k-1} B_i = 0 \tag{22}$$

Equation (20) implies that

$$\sum_{i=1}^{k-1} (A_i^T + B_i^T) \beta(\theta) + \sum_{i=1}^{k-1} B_i^T \delta^T(\theta, 0) = 0. \tag{23}$$

From equations (14) to (17) it results that

$$\gamma(\theta) = 0 \quad \text{for} \quad \theta \in [-\tau_k, 0]. \tag{24}$$

We denote

$$C = \sum_{i=1}^{k-1} A_i \tag{25}$$

and

$$D = \sum_{i=1}^{k-1} B_i. \tag{26}$$

Taking into account equations (22) to (26) we can write the set of equations (11) to (21) in the form

$$A^T \alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} = -W \tag{27}$$

$$2\alpha (C + D) + \beta(0) D = 0 \tag{28}$$

$$2\alpha (A_k + B_k) + \beta(0) B_k - \beta(-\tau_k) = 0 \tag{29}$$

$$A^T \beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta, 0) = 0 \tag{30}$$

$$(A_k + B_k)^T \beta(\theta) + B_k^T \delta^T(\theta, 0) - \delta(-\tau_k, \theta) = 0 \tag{31}$$

$$(C + D)^T \beta(\theta) + D^T \delta^T(\theta, 0) = 0 \tag{32}$$

$$\frac{\partial \delta(\theta, \sigma)}{\partial \theta} + \frac{\partial \delta(\theta, \sigma)}{\partial \sigma} = 0 \tag{33}$$

for $\theta \in [-\tau_k, 0]$, $\sigma \in [\theta, 0]$.

Because of relations (22) and (23), there does not exist equivalence between the sets of equations (11) to (21) and (27) to (33), but there exists an implication between them. The set of equations (27) to (33) is implied by the set of equations (11) to (21). The equivalence between those sets holds in case of $k = 2$.

Now we find a solution of the set of equations (27) to (33). We assume that the matrix D is not singular. From equation (28) we get

$$\beta(0) = -2\alpha\left(CD^{-1} + I\right) \tag{34}$$

and we put it into (27). After some calculations we obtain the relationship

$$\alpha G + G^T\alpha = -W \tag{35}$$

where

$$G = A - CD^{-1} - I \tag{36}$$

Matrix G should be negatively defined because matrix W is positively defined. From the formula (35) we can obtain the matrix α.

Now we take into account equations (30) and (32). We pre-multiply the equation (30) by D^T and we put into it the term $D^T\delta^T(\theta, 0)$, calculated from equation (32). After some calculations we have

$$\frac{d\beta(\theta)}{d\theta} = G^T\beta(\theta) \quad \text{for} \quad \theta \in [-\tau_k, 0] \tag{37}$$

where matrix G is given by formula (36).

The solution of the differential equation (37) is given by

$$\beta(\theta) = \exp(G^T(\theta + \tau_k)\beta(-\tau_k) \quad \text{for} \quad \theta \in [-\tau_k, 0]. \tag{38}$$

From equation (29) we obtain the initial condition of the differential equation (37)

$$\beta(-\tau_k) = \beta(0)B_k + 2\alpha(A_k + B_k). \tag{39}$$

We put into equation (39) the term (34). After calculation we get

$$\beta(-\tau_k) = 2\alpha\left(A_k - CD^{-1}B_k\right) \tag{40}$$

where α is a solution of equation (35).

Now we can obtain the solution of the differential equation (37), with the initial condition given by relation (40)

$$\beta(\theta) = 2\exp(G^T(\theta + \tau_k))\alpha\left(A_k - CD^{-1}B_k\right) \quad \text{for} \quad \theta \in [-\tau_k, 0]. \tag{41}$$

Now we determine the matrix $\delta(\theta, \sigma)$.

The solution of equation (33) is as below

$$\delta(\theta,\sigma)=\varphi(\theta-\sigma) \quad \text{for} \quad \theta\in[-\tau_k,0],\ \sigma\in[\theta,0] \tag{42}$$

where $\varphi\in C^1([-\tau_k,\tau_k],\mathbb{R}^{n\times n})$ and C^1 is a space of continuous functions with continuous derivative.

From equation (32) we get

$$\delta^T(\theta,0)=-\left(CD^{-1}+I\right)^T\beta(\theta) \quad \text{for} \quad \theta\in[-\tau_k,0]. \tag{43}$$

We put the term (43) into formula (31) and we get

$$(A_k+B_k)^T\beta(\theta)-B_k^T(CD^{-1}+I)^T\beta(\theta)-\delta(-\tau_k,\theta)=0. \tag{44}$$

After some computations we have

$$\delta(-\tau_k,\theta)=(A_k-CD^{-1}B_k)^T\beta(\theta) \quad \text{for} \quad \theta\in[-\tau_k,0]. \tag{45}$$

Taking into account relation (42), expression (45) takes the form

$$\varphi(-\tau_k-\theta)=(A_k-CD^{-1}B_k)^T\beta(\theta) \quad \text{for} \quad \theta\in[-\tau_k,0]. \tag{46}$$

So,

$$\varphi(\xi)=(A_k-CD^{-1}B_k)^T\beta(-\tau_k-\xi) \quad \text{for} \quad \xi\in[-\tau_k,0]. \tag{47}$$

Hence

$$\delta(\theta,\sigma)=\varphi(\theta-\sigma)=(A_k-CD^{-1}B_k)^T\beta(\sigma-\theta-\tau_k) \tag{48}$$
$$\text{for} \quad \theta\in[-\tau_k,0], \quad \sigma\in[\theta,0].$$

By putting into (48) relation (41) we get

$$\delta(\theta,\sigma)=2(A_k-CD^{-1}B_k)^T\exp(G^T(\sigma-\theta))\alpha\left(A_k-CD^{-1}B_k\right) \tag{49}$$
$$\text{for} \quad \theta\in[-\tau_k,0], \quad \sigma\in[\theta,0].$$

In this way we obtained all parameters of the Lyapunov functional (6).

4. A case of retarded type time delay system

Now we consider the problem in the case of retarded type time delay system. We write the set of equations (27) to (33) for this case. We obtain it by putting into equations the substitution $B_k=0$ and $D=0$. We obtain the following set of equations:

$$A^T\alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} = -W \tag{50}$$

$$2\alpha C = 0 \tag{51}$$

$$2\alpha A_k - \beta(-\tau_k) = 0 \tag{52}$$

$$A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta, 0) = 0 \tag{53}$$

$$A_k^T\beta(\theta) - \delta(-\tau_k, \theta) = 0 \tag{54}$$

$$C^T\beta(\theta) = 0 \tag{55}$$

$$\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} = 0 \tag{56}$$

for $\theta \in [-\tau_k, 0]$, $\sigma \in [\theta, 0]$.

One can see that this set of equations does not have a non-zero solution, therefore one needs another form of the Lyapunov functional for this case. Duda (2010) proposed a proper functional for the case of linear time invariant system with two lumped retarded type time delays and formulated the procedure of determination of its coefficients.

5. An example

Let us consider the system described by

$$\begin{cases} \frac{dx(t)}{dt} - d\frac{dx(t-\tau)}{dt} - e\frac{dx(t-r)}{dt} = ax(t) + bx(t-\tau) + cx(t-r) \\ x(t_0) = x_0 \\ x(t_0+\theta) = \Phi(\theta) \end{cases} \tag{57}$$

$$t \geq t_0,\ x(t) \in \mathbb{R}, \quad \theta \in [-r, 0), \quad a,\ b,\ c,\ d,\ e \in \mathbb{R}, \quad 0 \leq \tau \leq r.$$

We introduce a new variable

$$y(t) = x(t) - dx(t-\tau) - ex(t-r) \quad \text{for} \quad t \geq t_0. \tag{58}$$

Equation (57) takes the form

$$\begin{cases} \frac{dy(t)}{dt} = ay(t) + (b+d)x(t-\tau) + (c+e)x(t-r) \\ y(t) = x(t) - dx(t-\tau) - ex(t-r) \\ y(t_0) = x_0 - d\Phi(-\tau) - e\Phi(-r) \\ x(t_0+\theta) = \Phi(\theta) \end{cases} \tag{59}$$

$$t \geq t_0, \quad y(t) \in \mathbb{R}, \quad \theta \in [-r, 0), \quad a,\ b,\ c,\ d,\ e \in \mathbb{R}, \quad 0 \leq \tau \leq r.$$

The Lyapunov functional is defined by the formula

$$V(S(t)) = \alpha y^2(t) + \int_{-r}^{0} y(t)\beta(\theta)x(t+\theta)d\theta + \tag{60}$$

$$+ \int_{-r}^{0} \gamma(\theta)x^2(t+\theta)d\theta + \int_{-r}^{0}\int_{\theta}^{0} \delta(\theta,\sigma)x(t+\theta)x(t+\sigma)d\theta d\sigma.$$

We obtain coefficients of the functional (60) as below.

According to equation (24)

$$\gamma(\theta) = 0 \quad for \quad \theta \in [-r, 0]. \tag{61}$$

Equations (35) and (36) take the form

$$2g\alpha = -w \tag{62}$$

$$g = a - \frac{b}{d} - 1 \tag{63}$$

where $w > 0$ and $g < 0$.

This is, so because the Lyapunov functional is positively defined and its derivative on the trajectory of the system (59) is negatively defined.

From equation (62) we obtain

$$\alpha = -\frac{w}{2g}. \tag{64}$$

According to equation (41)

$$\beta(\theta) = 2\alpha p \exp(g(\theta + r)) = -\frac{wp}{g}\exp(g(\theta + r)) \quad for \quad \theta \in [-r, 0] \tag{65}$$

where

$$p = c - \frac{be}{d}. \tag{66}$$

From the formula (49) we obtain

$$\delta(\theta,\sigma) = 2\alpha p^2 \exp(g(\sigma - \theta)) = -\frac{wp^2}{g}\exp(g(\sigma - \theta)) \tag{67}$$

for $\theta \in [-r, 0], \quad \sigma \in [\theta, 0]$.

6. Conclusions

The paper presents the procedure of determining the coefficients of the Lyapunov functional, given by the formula (6), for the linear system with k-non-commensurate neutral type time delays, described by equation (1). This paper extends the method presented by Repin to the systems with k non-commensurate neutral type time delays. The method presented allows for achieving the analytical formula on the factors occurring in the Lyapunov functional, which can be used to examine the stability and in the process of parametric optimization to designate the square index of the quality given by the formula (10).

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