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Simple conditions for robust stability of positive discrete-time linear systems with delays*

by

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Abstract: The paper is devoted to the problem of robust stability of positive linear discrete-time systems with delays in the case of structured perturbations of state matrices. Simple new necessary and sufficient conditions for robust stability in the general case and in the case of system with linear uncertainty structure are established for two sub-cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices. It is shown that robust stability of the positive discrete-time linear system with delays is equivalent to: 1) robust stability of the corresponding positive system without delays of the same order as time-delay system - in the general case, 2) asymptotic stability of finite family of the positive vertex systems without delays - in the case of a linear unity rank uncertainty structure, 3) asymptotic stability of only one positive vertex system without delays – in the case of a linear uncertainty structure with non-negative perturbation matrices.

Keywords: stability, robust stability, linear system, positive, discrete-time, delays, linear uncertainty, interval system.

1. Introduction

A dynamical system is called positive if any trajectory of the system starting form non-negative initial states remains forever non-negative for non-negative controls. Examples of positive systems can be found in engineering, biology and medicine, economics, management science, social sciences, etc. An overview of state of the art in positive systems theory is given in the monographs of Farina and Rinaldi (2000) and Kaczorek (2002).

The problem of robust stability of continuous-time and discrete-time linear systems was considered in the monographs of Ackermann et al. (1995), Bhattacharyya, Chapellat and Keel (1995), Białas (2002), Busłowicz (1997), for example.

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Conditions for stability and robust stability of positive discrete-time linear systems with delays with structured perturbations of state matrices were given in Busłowicz (2005, 2007), Busłowicz, Kaczorek (2004a, 2004b), and Kaczorek (2004, 2006). These conditions are expressed in terms of the equivalent positive discrete-time system without delays of the order $(h+1)n$, where n is the order of the system with h delays.

Recently, it was shown that checking of asymptotic stability of positive discrete-time linear systems with delays can be reduced to checking of asymptotic stability of the corresponding positive discrete-time system without delays of the same order as the time-delays system. Conditions for stability (or robust stability) were given in Busłowicz, Kaczorek (2004c, 2005) for systems with one pure delay, in Busłowicz (2008a) and Hmamed, Benzaouia, Rami, Tadeo (2007) for systems with one delay, and in Busłowicz (2008b) for systems with multiple delays.

The main purpose of the paper is to give simple new necessary and sufficient conditions for robust stability of linear positive discrete-time systems with delays with structured perturbations of state matrices in a general case and in the case of linear uncertainty structure in two sub-cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices.

The results will be obtained by extending the asymptotic stability conditions, given in Busłowicz (2008b).

In the paper the following notations will be used: $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with non-negative entries and $\Re_+^n = \Re_+^{n\times 1}$; Z_+ - the set of non-negative integers; I_n - the $n\times n$ identity matrix; a vector $x\in\Re^n$ will be called strictly positive (strictly negative) and denoted by $x>0$ $(x<0)$ if all its entries are positive (negative).

2. Problem formulation

Consider an uncertain positive discrete-time linear system with delays described by the homogeneous equation

$$x_{i+1} = \sum_{k=0}^{h} A_k(q)x_{i-k}, \quad q\in Q, \quad i\in Z_+, \tag{1}$$

with the initial conditions x_{-k}, $k=0,1,...,h$, where h is a positive integer,

$$A_k(q)\in\Re_+^{n\times n}, \quad \forall q\in Q(k=0,1,...,h), \tag{2}$$

$q=[q_1,q_2,...,q_m]^{\mathrm{T}}$ is the vector of uncertain physical parameters of the system (1) and

$$Q=\{q: q_r\in[q_r^-,\ q_r^+],\ \ r=1,2,...,m\} \tag{3}$$

with $q_r^-\le 0$, $q_r^+\ge 0$ $(r=1,2,...,m)$ is the value set of uncertain parameters.

In Busłowicz (2007) the following positive discrete-time linear system with delays has been considered

$$x_{i+1} = \sum_{k=0}^{h} A_k(q_k) x_{i-k}, \quad q_k \in Q_k, \quad i \in Z_+, \tag{4}$$

where $q_k = [q_{k1}, q_{k2}, ..., q_{km_k}]^{\mathrm{T}} \in Q_k$ is the k-th ($k = 0, 1, ..., h$) sub-vector of uncertain parameters q_{k1}, q_{k2},..., q_{km_k}, and $Q_k = \{q_k : q_{kr} \in [q_{kr}^-,\ q_{kr}^+],$ $r = 1, 2, ..., m_k\}$ is the value set of these parameters.

It is easy to see that the state equation (4) can be written in the form (1) with $q \in Q$ and $q = [q_0^{\mathrm{T}}, q_1^{\mathrm{T}}, ..., q_h^{\mathrm{T}}]^{\mathrm{T}}$, $Q = Q_0 \times Q_1 \times \cdots \times Q_h$. Hence, the equation (1) has a more general form than the equation (4).

By generalization of the positivity condition of a discrete-time linear system with delays without uncertain parameters (Kaczorek, 2004, 2006) to the case of uncertain parameters, one obtains the following definition and lemma.

Definition 1 *The system* (1) *will be called positive (internally) if for any* $q \in Q$ *the following condition holds:* $x_i \in \Re_+^n$ *for all* $i \in Z_+$ *and for any initial conditions* $x_{-k} \in \Re_+^n$, $k = 0, 1, ..., h$.

Lemma 1 *The system* (1) *is positive if and only if condition* (2) *holds.*

We will assume in this paper that all the entries of matrices $A_k(q)$ ($k = 0, 1, ..., h$) are continuous functions of uncertain parameters, non-linear or linear.

In the case of linear uncertainty structure, all the entries of $A_k(q)$ are linear continuous functions of uncertain parameters. Therefore, we may write

$$A_k(q) = A_{k0} + \sum_{r=1}^{m} q_r E_{kr}, \quad k = 0, 1, ..., h, \tag{5}$$

where $A_{k0} \in \Re_+^{n\times n}$ and $E_{kr} \in \Re^{n\times n}$ ($k = 0, 1, ..., h,\ r = 1, 2, ..., m$) are the nominal and the perturbation matrices, respectively.

Definition 2 *The system* (1) *will be called a system with linear unity rank uncertainty structure if the following conditions hold*

$$\mathit{rank}\ \tilde{E}_r = 1, \quad r = 1, 2, ..., m, \tag{6}$$

where

$$\tilde{E}_r = \sum_{k=0}^{h} E_{kr}. \tag{6a}$$

The discrete-time system (1) has a linear uncertainty structure with non-negative perturbation matrices if

$$E_{kr} \in \Re_+^{n\times n}, \quad k = 0, 1, ..., h, \quad r = 1, 2, ..., m. \tag{7}$$

The positive system (1) is robustly stable if and only if all roots $z_1(q)$, $z_2(q)$, ... $z_{\tilde{n}}(q)$ of the characteristic equation

$$\det\left(z^{h+1}I_n - \sum_{k=0}^{h} A_k(q)z^{h-k}\right) = 0 \tag{8}$$

satisfy the conditions $|z_i(q)| < 1$ for all $q \in Q$ and $i = 1, 2, ..., \tilde{n} = (h+1)n$ (Busłowicz, 2007).

Necessary and sufficient conditions for robust stability of the positive system (4) have been given by Busłowicz (2007). These conditions are expressed in terms of the equivalent discrete-time system without delays

$$\tilde{x}_{i+1} = \tilde{A}(q)\tilde{x}_i, \quad q \in Q, \tag{9}$$

where $\tilde{x}_i = [x_i^{\mathrm{T}}, x_{i-1}^{\mathrm{T}}, \cdots, x_{i-h}^{\mathrm{T}}]^{\mathrm{T}} \in \Re_+^{\tilde{n}}$ with $\tilde{n} = (h+1)n$ and

$$\tilde{A}(q) = \begin{bmatrix} A_0(q_0) & A_1(q_1) & \cdots & A_h(q_h) \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times\tilde{n}}. \tag{10}$$

The results of Busłowicz (2007) can be easily reformulated for the system (1).

The aim of this paper is to give new, simple necessary and sufficient conditions for robust stability of positive discrete-time linear systems with delays (1) in the general case and in the case of systems with a linear uncertainty structure in two sub-cases:

1) unity rank uncertainty structure (the conditions (6) hold),
2) non-negative perturbation matrices (the conditions (7) hold, satisfaction of (6) is not necessary).

First, we show that robust stability of the positive discrete-time system (1) with delays is equivalent to robust stability of the corresponding positive discrete-time system without delays of the order n, i.e. of the order extremely less than the order of the equivalent system (9) (equal to $\tilde{n} = (h+1)n$). Next, we give simple conditions for robust stability.

3. Robust stability of linear positive discrete-time systems with delays

The following theorems and lemma have been proved in Busłowicz (2008b).

Theorem 1 *The positive discrete-time linear system with delays*

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k}, \quad i \in Z_+, \tag{11}$$

is asymptotically stable (independent of delays) if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ *(i.e.* $\lambda > 0$*) such that* $[A - I_n]\lambda < 0$ *or, equivalently, the positive system without delays*

$$x_{i+1} = Ax_i, \quad i \in Z_+, \tag{12}$$

is asymptotically stable, where

$$A = \sum_{k=0}^{h} A_k \in \Re_+^{n \times n}. \tag{13}$$

THEOREM 2 *The positive discrete-time system* (11) *with delays is asymptotically stable (independent of delays) if and only if one of the following equivalent conditions holds:*

1) *eigenvalues* $z_1, z_2, \cdots, z_n$ *of the matrix* A *defined by* (13) *have moduli less than 1,*
2) *all the leading principal minors of the matrix* $S = I_n - A$ *are positive,*
3) *all the coefficients of the characteristic polynomial of the matrix* $-S = A - I_n$ *are positive,*
4) $\rho(A) < 1$*, where* $\rho(A)$ *is the spectral radius of the matrix* A*.*

LEMMA 2 *The positive discrete-time system* (11) *with delays is unstable if at least one diagonal entry of matrix* A *is greater than 1.*

From Lemma 2 and the positivity condition $A_k \in \Re_+^{n \times n}$ $(k = 0, 1, ..., h)$ we have the following remark.

REMARK 1 *The positive system* (11) *is unstable if at least one diagonal entry of any matrix* A_k $(k = 0, 1, ..., h)$ *is greater than 1.*

By comparison of conditions of asymptotic stability of the standard (i.e. non-positive) discrete-time linear system with delays and the conditions given in Theorems 1 and 2 for positive systems, we obtain the following important remark.

REMARK 2 *The standard positive discrete-time linear system with delays* (11) *is asymptotically stable if and only if all roots* z_i $(i = 1, 2, ..., \tilde{n} = (h + 1)n)$ *of the characteristic polynomial*

$$w(z) = \det \left(z^{h+1} I_n - \sum_{k=0}^{h} A_k z^{h-k} \right) \tag{14}$$

have moduli less than 1, whereas this system with $A_k \in \Re_+^{n \times n}$ $(k = 0, 1, ..., h)$ *is asymptotically stable if and only if all eigenvalues of the matrix* (13) *have absolute values less than 1.*

3.1. Robust stability in the general case

By generalisation of Theorem 1 to the case of the system (1) with uncertain parameters one obtains the following theorem.

THEOREM 3 *The positive discrete-time linear system with delays* (1) *is robustly stable (independent of delays) if and only if the positive discrete-time system without delays*

$$x_{i+1} = A(q)x_i, \quad q \in Q, \quad i \in Z_+, \tag{15}$$

is robustly stable, where

$$A(q) = \sum_{k=0}^{h} A_k(q) \in \Re_+^{n\times n}. \tag{16}$$

By generalisation of Theorem 2 and Lemma 2 (with Remark 1) to the system (1) with uncertain parameters we obtain the following theorem and lemma.

THEOREM 4 *The positive discrete-time system with delays* (1) *is robustly stable if and only if the following equivalent conditions hold:*

1) all the leading principal minors $\Delta_i(q)$ $(i = 1, 2, ..., n)$ *of the matrix* $S(q) = I_n - A(q)$ *are positive for all* $q \in Q$, *i.e.*

$$\min_{q\in Q} \Delta_i(q) > 0, \quad i = 1, 2, ..., n, \tag{17}$$

2) all the coefficients of the characteristic polynomial of the matrix $-S(q) = A(q) - I_n$, *of the form*

$$w(z, q) = \det(zI_n + S(q)) = z^n + \sum_{i=0}^{n-1} a_i(q)z^i, \tag{18}$$

are positive for all $q \in Q$, *i.e.*

$$\min_{q\in Q} a_i(q) > 0, \quad i = 0, 1, ..., n-1. \tag{19}$$

LEMMA 3 *The positive discrete-time system* (1) *with delays is not robustly stable if there is such a* $q \in Q$ *that at least one diagonal entry of any matrix* $A_k(q)$ $(k = 0, 1, ..., h)$ *is greater than 1.*

The conditions (17) and (19) can be checked by using computer programs for minimization with constraints of real multivariable functions.

EXAMPLE 1 Consider the positive discrete-time system (1) for $n = 2$, $h = 1$, $m = 2$ with the matrices

$$A_0(q) = \begin{bmatrix} 0.1 + q_1^2 & 0.3 - q_2^2 \\ 0 & 0.4 - q_1 \end{bmatrix}, A_1(q) = \begin{bmatrix} 0.4 + q_2 & 0 \\ 0.25 + q_1 - q_2 & 0.2 - q_2^2 \end{bmatrix}, \tag{20}$$

where $q \in Q$ and

$$Q = \{q = [q_1,\ q_2]^{\mathrm{T}} : q_r \in [-0.1,\ 0.1],\ \ r = 1,\ 2\}. \tag{21}$$

From (20) and (21) it follows that

$$A(q) = A_0(q) + A_1(q) = \begin{bmatrix} 0.5 + q_1^2 + q_2 & 0.3 - q_2^2 \\ 0.25 + q_1 - q_2 & 0.6 - q_1 - q_2^2 \end{bmatrix} \tag{22}$$

has non-negative all entries for all $q \in Q$ and the matrix $S(q) = I_2 - A(q)$ has the form

$$S(q) = \begin{bmatrix} 0.5 - q_1^2 - q_2 & -0.3 + q_2^2 \\ -0.25 - q_1 + q_2 & 0.4 + q_1 + q_2^2 \end{bmatrix}. \tag{23}$$

Computing the leading principal minors of (23) we obtain: $\Delta_1(q) = 0.5 - q_1^2 - q_2$, $\Delta_2(q) = \det S(q)$ and $\min_{q \in Q} \Delta_1(q) = 0.39 > 0$, $\min_{q \in Q} \Delta_2(q) = 0.102 > 0$. This means that all the leading principal minors of (23) are positive for all $q \in Q$ and the system is robustly stable, according to condition 1) of Theorem 4.

3.2. Robust stability of systems with a linear unity rank uncertainty structure

In the case of a linear uncertainty structure, using (16) and (5) we can write

$$A(q) = A_0 + E(q), \tag{24}$$

where

$$A_0 = \sum_{k=0}^{h} A_{k0}, \quad E(q) = \sum_{r=1}^{m} q_r \tilde{E}_r, \tag{25}$$

and $\tilde{E}_r$ is defined by (6a).

LEMMA 4 *Asymptotic stability of the positive nominal system*

$$x_{i+1} = A_0 x_i \tag{26}$$

is necessary for robust stability of the positive system (1) *with a linear uncertainty structure.*

Proof. Follows from Theorem 3 for $q = 0$ and equality $A(0) = A_0$. ■

To the stability analysis of the positive system (26), Theorem 2 for $A = A_0$ can be applied.

From the above, Lemma 2 and Remark 1 we have the following lemma.

LEMMA 5 *The positive discrete-time system* (1) *with a linear uncertainty structure is not robustly stable if at least one diagonal entry of any matrix* A_{k0} $(k = 0, 1, ..., h)$ *in* (5) *is greater than 1.*

Now we consider the positive system (1) with linear unity rank uncertainty structure (the condition (6) holds). In this case the matrix (24) has linear unity rank uncertainty structure and all coefficients of the polynomial (18) are real multilinear functions of uncertain parameters.

Let us denote by $\bar{q}_1, \bar{q}_2, ..., \bar{q}_K$ $(K = 2^m)$, where $\bar{q}_k = [\hat{q}_1, \hat{q}_2, ..., \hat{q}_m]^T$ with $\hat{q}_r = q_r^-$ or $\hat{q}_r = q_r^+$, $r = 1, 2, ..., m$, the vertices of hiperrectangle (3).

Moreover, by $V_k = A(\bar{q}_k)$, $k = 1, 2, ..., K$, denote the vertex matrices of the family of non-negative matrices $\{A(q) : q \in Q\}$, where $A(q)$ has the form (24). These matrices correspond to the vertices of the set (3).

THEOREM 5 *The positive discrete-time system* (1) *with linear unity rank uncertainty structure is robustly stable if and only if all the positive vertex systems without delays*

$$x_{i+1} = V_k x_i, \quad k = 1, 2, ..., K, \tag{27}$$

are asymptotically stable, i.e. the conditions of Theorem 2 are satisfied for $A = V_k$ *and for all* $k = 1, 2, ..., K$.

Proof. Necessity is obvious because the systems (27) belong to the family (15) of positive systems.

The proof of sufficiency is based on the following observation: if the system (1) has linear unity rank uncertainty structure (the conditions (6) hold) then the coefficients $a_i(q)$, $i = 0, 1, ..., n-1$, of (18) are real multilinear functions of uncertain parameters q_r, $r = 1, 2, ..., m$, and therefore

$$\min_{q \in Q} a_i(q) = \min_k a_i(\bar{q}_k), \quad i = 0, 1, ..., n-1. \tag{28}$$

From the condition 3) of Theorem 2 it follows that if the family (27) of the positive systems is asymptotically stable, then all coefficients of the characteristic polynomials of the matrices $-S_k = V_k - I_n$, $k = 1, 2, ..., K$, are positive, i.e.

$$a_i(\bar{q}_k) > 0, \quad i = 0, 1, ..., n-1, \quad k = 1, 2, ..., K. \tag{29}$$

Hence, $\min_k a_i(\bar{q}_k) > 0$ for $i = 0, 1, ..., n-1$, and by (28),

$$\min_{q \in Q} a_i(q) > 0, \quad i = 0, 1, ..., n-1. \tag{30}$$

This means that all coefficients of the polynomial (18) are positive for all $q \in Q$, and by condition 2) of Theorem 4, the positive system (1) is robustly stable. ∎

To the analysis of asymptotic stability of the positive systems (27) we can apply Theorem 2 assuming $V_k = A(\bar{q}_k)$ for $k = 1, 2, ..., K$, instead of the matrix A.

Example 2 Check robust stability of the positive discrete-time system (1) with $n = 2$, $h = 2$, $m = 2$ and matrices $A_k(q)$, $k = 0, 1, 2$, of the form (5) with

$$A_{00} = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix},\ A_{10} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix},\ A_{20} = \begin{bmatrix} 0 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \tag{31a}$$

$$E_{01} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{31b}$$

$$E_{02} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{31c}$$

where $q \in Q$, with

$$Q = \{q = [q_1, q_2]^{\mathrm{T}} : q_r \in [-0.1, 0.1],\ r = 1, 2\}. \tag{32}$$

It is easy to check that condition (2) holds. Hence, the system (1) with the matrices (5), (31) is positive. Moreover, it is easy to see that this system has a linear unity rank uncertainty structure (the conditions (6) hold) and the nominal system (26) is asymptotically stable because all the leading principal minors of the matrix

$$S_0 = I_2 - A_0 = \begin{bmatrix} 0.7 & -0.6 \\ -0.3 & 0.7 \end{bmatrix},$$

are positive.

We apply Theorem 5 to perform the robust stability analysis.

The set (32) of $m = 2$ uncertain parameters has $K = 2^m = 4$ vertices. Hence, there are $K = 4$ vertex systems (27). Asymptotic stability of the vertex systems is necessary and sufficient for robust stability of the system under consideration.

Computing the vertices of the set of uncertain parameters (32), the vertex matrices $V_k = A(\bar{q}_k)$, and the matrices $S_k = I_2 - V_k$, $k = 1, 2, ..., 4$, one obtains

$$\bar{q}_1 = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix},\ \bar{q}_2 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix},\ \bar{q}_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ \bar{q}_4 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 0.8 & -0.1 \\ -0.3 & 0.8 \end{bmatrix},\ S_2 = \begin{bmatrix} 0.8 & -0.7 \\ -0.3 & 0.6 \end{bmatrix}, \tag{33a}$$

$$S_3 = \begin{bmatrix} 0.6 & -1.1 \\ -0.3 & 0.6 \end{bmatrix},\ S_4 = \begin{bmatrix} 0.6 & -0.5 \\ -0.3 & 0.8 \end{bmatrix}. \tag{33b}$$

It is easy to check that all the leading principal minors of matrices (33) are positive. This means that all the positive vertex systems (27) are asymptotically stable, according to condition 2) of Theorem 2. Hence, from Theorem 5 it follows that the system is robustly stable.

We obtain the same result from Theorem 4, because all the leading principal minors of the matrix

$$S(q) = I_2 - A(q) = \begin{bmatrix} 0.7 - q_1 & -0.6 - 2q_1 - 3q_2 \\ -0.3 & 0.7 - q_2 \end{bmatrix} \tag{34}$$

are positive for all $q \in Q$.

3.3. Robust stability of systems with a linear uncertainty structure and non-negative perturbation matrices

Recall that real $n \times n$ interval matrix $[A^-, A^+]$ is such a set of real $n \times n$ matrices $A = [a_{ij}]$ that $a_{ij}^- \leq a_{ij} \leq a_{ij}^+$, $i, j = 1, 2, ..., n$, where $A^- = [a_{ij}^-]$, $A^+ = [a_{ij}^+]$.

Consider the positive discrete-time system (1) with the state matrices of the form (5) satisfying the condition (7).

In this case $q_r E_{kr} \in [q_r^- E_{kr}, q_{kr}^+ E_r]$ for all $q_r \in [q_r^-, q_r^+]$. This means that $A_k(q) \in [A_k^-, A_k^+] \subset \Re_+^{n \times n}$ for all $q \in Q$ and $k = 0, 1, ..., h$, where

$$A_k^- = A_{k0} + \sum_{r=1}^{m} q_r^- E_{kr}, \quad A_k^+ = A_{k0} + \sum_{r=1}^{m} q_r^+ E_{kr}. \tag{35}$$

From (24), (25) and (35) we have that $A(q) \in [A^-,\ A^+] \subset \Re_+^{n \times n}$ for all $q \in Q$, where

$$A^- = \sum_{k=0}^{h} A_k^-, \quad A^+ = \sum_{k=0}^{h} A_k^+. \tag{36}$$

It is easy to see that the interval matrix $[A^-, A^+]$ is non-negative if and only if $A^- \in \Re_+^{n \times n}$.

The matrices (36) can be computed form the formulas

$$A^- = A_0 + E^-, \quad A^+ = A_0 + E^+, \tag{37}$$

where A_0 has the form given in (25) and

$$E^- = \sum_{r=1}^{m} q_r^- \tilde{E}_r, \quad E^+ = \sum_{r=1}^{m} q_r^+ \tilde{E}_r, \tag{38}$$

where $\tilde{E}_r$ is defined by (6a).

From the above and Theorem 3 it follows that robust stability of the positive discrete-time system (1) with linear uncertainty structure and non-negative perturbation matrices is equivalent to robust stability of the positive discrete-time interval system without delays

$$x_{i+1} = Ax_i, \quad A \in [A^-,\ A^+] \subset \Re_+^{n \times n}. \tag{39}$$

Bhattacharyya, Chapellat and Keel (1995) showed that robust stability of the positive interval discrete-time system (39) is equivalent to asymptotic stability of the positive system

$$x_{i+1} = A^+ x_i, \quad i \in Z_+. \tag{40}$$

Hence, we have the following theorem and lemma.

THEOREM 6 *The positive discrete-time system* (1) *with a linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if the positive discrete-time system* (40) *is asymptotically stable, where* A^+ *has the form given in* (36) *(or* (37)*).*

LEMMA 6 *The positive discrete-time system* (1) *with a linear uncertainty structure and non-negative perturbation matrices is not robustly stable if at least one diagonal entry of the matrix* A^+ *is greater than 1.*

From Theorem 6 it follows that robust stability of the positive discrete-time system (1) with delays with a linear uncertainty structure and non-negative perturbation matrices is equivalent to asymptotic stability of this system for $q = q^+ = [q_1^+, q_2^+, ..., q_m^+]^{\mathrm{T}}$, i.e. of the system

$$x_{i+1} = \sum_{k=0}^{h} A_k^+ x_{i-k}, \quad i \in Z_+,$$

where A_k^+ has the form given in (35).

EXAMPLE 3 Check robust stability of the positive system considered in Example 2 with

$$E_{02} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{41}$$

and the remaining matrices of the forms given in (31).

The system under consideration is a positive system with linear uncertainty structure with non-negative perturbation matrices. Therefore, we use Theorem 6 for robust stability analysis.

Computing the matrix A^+ from (37), (38) and the matrix $S^+ = I_2 - A^+$ we obtain

$$S^+ = \begin{bmatrix} 0.6 & -1.1 \\ -0.3 & 0.4 \end{bmatrix}. \tag{42}$$

Matrix (42) has non-positive leading principal minor $\Delta_2^+ = \det S^+ = -0.09$. This means that the positive system (40) is not asymptotically stable and the system is not robustly stable, according to Theorem 6.

4. Concluding remarks

New, simple necessary and sufficient conditions have been given for robust stability of the positive discrete-time linear system with delays (1) in the general case and in the case of system with a linear uncertainty structure in two subcases: 1) unity rank uncertainty structure (the conditions (6) hold), 2) non-negative perturbation matrices (the conditions (7) hold, satisfaction of (6) is not necessary).

It has been shown that:

- robust stability of the positive system (1) with delays is equivalent to the robust stability of the corresponding positive discrete-time system without delays (15) (Theorem 3),
- the positive system (1) with delays with a linear unity rank uncertainty structure is robustly stable if and only if the positive vertex systems (27) are asymptotically stable (Theorem 5),
- the positive system (1) with delays with a linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if the system (40) is asymptotically stable (Theorem 6).

The proposed conditions for robust stability have been obtained by extension of asymptotic stability conditions given in Busłowicz (2008b) to the case of positive systems with uncertain parameters.

Similar results for the positive continuous-time linear system with delays were given in Busłowicz (2010).

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