Control and Cybernetics

vol. **39** (2010) No. 3

Guaranteed control policy with arbitrary set of correction points for linear-quadratic system with delay*

by

Kil To Chong[1], Olga Kostyukova[2] and Mariya Kurdina[2]

[1] Institute of Information and Communication, Chonbuk National University
Chonju, 561-765, Korea

[2] Institute of Mathematics, National Academy of Sciences of Belarus
Minsk 220072, Belarus

e-mail: kitchong@moac.chonbuk.ac.kr, kostyukova@im.bas-net.by

Abstract: For continuous, uncertain, linear quadratic control system with delayed input, we consider a min-max control policy in which the elements of feedback are present. The feedback is introduced into control optimization by allowing a control to be corrected at a given set of correction points from the control interval. This helps to overcome the feasibility difficulties that arise with standard min-max techniques. We show that construction of the optimal policy involves a sequence of min-max optimizations formulated as dynamic programs that do not yield simple analytical solutions. That is why the paper is mainly focused on construction and justification of suboptimal control policy that can be effectively implemented. Simulated examples demonstrate the proposed approach.

Keywords: delayed system, guaranteed optimal control policy, approximative control policy.

1. Introduction

The existence of uncertain systems with delay is evident in a broad range of fields. Certainly, many realistic objects in physics, biology, chemistry, economics are modeled by uncertain systems of differential equations with delayed inputs. Effective solution of optimal control problems for such complex realistic system is connected with essential difficulties that are caused by the presence of system uncertainties, unknown external disturbances and delays in system devices. Conventional control theory with many ideal assumptions, such as deterministic system behavior and nondelayed sensing and actuation, is not applicable to these complex realistic systems. In this respect, it is important to develop new methods that could manipulate properly the system uncertainties and delays

*Submitted: January 2007; Accepted: October 2009.

that occur while exchanging data among control system components (sensors, controller, actuator, etc.).

In recent years, control problems for linear systems with delay have been considered in many publications depending on the delay type, specific system equations, performance index, etc. (see Basin and Rodriguez-Gonzalez, 2005; Boukas and Al-Muthairi, 2006). A comprehensive review of theory and algorithms for time-delay systems can be found in Richard (2003).

Control system design that can handle model uncertainties and perturbations has been one of the most challenging problems and received considerable attention from control engineers and scientists (see Mahmoud, 2000). In control engineering practice, it is often desirable to design a control, which does not only satisfy state-control constraints but also guarantees an adequate level of performance.

I. One approach to this problem is the so-called *guaranteed cost control*. In Kim (2000), Shi, Boukas, Shi and Agarwal (2003), Yu and Gao (2001), by using linear matrix inequality (LMI) techniques, the optimal guaranteed cost control problem is studied for different classes of linear systems with delay under norm-bounded uncertainties and given quadratic cost function. Sufficient conditions for existence of linear guaranteed cost control laws are presented. The problems of designing the optimal guaranteed cost controllers are reduced to corresponding convex optimization problems with LMI constraints.

Since, in this formulation, control law is looked for in a prescribed linear form, this leads to strong sufficient conditions for existence of such control and often implies the situations when there does not exist the desirable control. Note that even when such control exists, this method requires solving complicated optimization problems with LMI constraints to construct a controller.

There are several other approaches to handle control optimization problems with uncertainties, see Lee and Yu (1997).

II. If a solution of a problem has the best value of a cost functional in the problem under the worst actual disturbance, we get the so-called *Open-loop worst-case optimal control (OLWOC).* This approach relies on min-max optimization of performance. In comparison with optimization problems arising in other approaches, this problem is not very complicated. However, this approach, as well as approach I, does not include the fact that more information about the states becomes available as time progresses. This formulation optimizes a single control over all possible disturbances. Obviously, this may cause feasibility difficulties that are likely to arise with this formulation, see Scokaert and Mayne (1998).

III. If, by forming an optimization problem, we include a possibility to correct a control depending on the measurements of the states, we construct a *Closed-loop worst-case optimal control (CLWOC).* The solution of the optimization problem results in the so-called worst-case optimal control policy

$$\pi = \{u_0(\cdot|x), u_1(\cdot|x), \ldots, u_m(\cdot|x)\}$$

consisting of control laws $u_i(\cdot|x)$ for all time intervals $t \in T_i = [t_i, t_{i+1})$ from the control interval $[0, t_{m+1}]$. This often leads to improved performance compared to OLWOC schemes. The method also avoids the unfeasibility problems that may result from the use of OLWOC formulations. The price that must be paid for these benefits is that the computational demands of the feedback min-max algorithm for the problem may be very high.

There are several possibilities to solve an optimization problem in the third approach for discrete systems, see Bemporad, Borelli and Morari (2003), Kerrigan and Maciejowski (2004), Kothare, Balakrishnan and Morari (1996), Magni, De Nicolao, Scattolini and Allgöwer (2002), Scokaert and Mayne (1998), Vanderberghe, Boyd and Nouralishahi (2002). In all these approaches, resulting methods turn out to be computationally demanding, either because the problem, which is solved on-line, has usually rather high dimensionality, or because the method suffers from the curse of dimensionality due to storing huge amounts of information if the main part of computations is made off-line. Since determination of a control policy is usually prohibitively difficult, research has focused on simplifying the closed-loop worst-case problem (Lee and Yu, 1997).

In this paper, we study a continuous linear-quadratic optimal control problem subject to additive uncertainties, bounded in energy sense, that is a generalization of a particular case that was earlier investigated in Kostina and Kostyukova (2006). There, the control problem without delay ($h = 0$) and only one correction point ($m = 1$) was studied. For this special case, optimal guaranteed control policy was proposed and justified.

Now we consider a more realistic situation. Namely, we take into account the computation delay $h > 0$ that is always present in a real control process and allow for correction of the control laws at $m \geq 1$ fixed intermediate time points $t_i \in [0, t_*]$, $i = 1, \ldots, m$, depending on the realized system state $z(t_i)$ at the time moment t_i, and a known control $u(t), t \in [t_i - h, t_i]$, that was constructed at the moment t_{i-1}. The proposed formulation can be viewed as a variant of Model Predictive Control scheme (with shrinking horizon), operating somewhere between the discrete and continuous time. We prove the existence of an optimal control policy and justify theoretical relations that determine this policy.

Construction of the optimal policy involves a sequence of min-max optimizations formulated as dynamic programs. Since the min-max problem at each stage does not yield a simple analytical solution, it must be solved numerically. Since such approach is numerically demanding and suffers from the curse of dimensionality, we propose to consider an approximative (suboptimal) policy. For this policy, we present not only a conceptual scheme but give detailed constructive rules of its implementation.

We show that computation of the approximative policy is equivalent to solving a corresponding convex mathematical programming (MP) problem with m decision variables. The MP problem may be solved on-line.

The paper is organized as follows. In Section 2, we consider a terminal linear-quadratic optimal control problem with delay in the presence of an additive, unknown, but bounded, uncertainty. For this system, we define the problem of constructing a guaranteed optimal strategy, which is allowed to be corrected at m fixed intermediate time moments. In Section 3, we present theoretical relations, defining an optimal control policy π^0, which solves the optimization problem. In Section 4, we introduce and justify an approximative policy and give simple rules for its construction. Another type of control policy is proposed in Section 5. Results of numerical experiments are presented in Section 6.

Throughout the paper we will use the following notations: $\lambda_{\max}(S)$ to denote the maximal eigenvalue of a matrix S, and $||y||_S$ to denote the weighted norm with some positive definite matrix S: $||y||^2_S := y^T Sy$, $\|y\|^2_2 = y^T y$.

2. Problem statement

In this section, we consider a terminal linear control system with delay, subject to unknown but bounded disturbances. Let dynamics of an object be defined by the differential equation

$$\dot{z}(t) = Az(t) + bu(t-h) + gw(t), \tag{1}$$

$$z(0) = z_0, u(t) = v^*(t), t \in [-h, 0],$$

$$\text{rank}(b, Ab, ..., A^{n-1}b) = n, \ \text{rank}(g, Ag, ..., A^{n-1}g) = n. \tag{2}$$

Here, $z(t) \in R^n$ denotes the state of the system, $u(t) \in R$ denotes control in the time moment $t \geq 0$, $h > 0$ is a delay in control; initial system state z_0 and initial control $v^*(t), t \in [-h, 0]$, are supposed to be given, $w(\cdot) = (w(t),\ t \in [0, t_*])$ is an unknown in advance disturbance from a bounded set $\Omega \subset L_2[0, t_*]$, which will be defined later, $A \in R^{n\times n}$, b, $g \in R^n$ are given matrix and vectors.

Let us denote by $z(t|u_{t-h}(\cdot), w_t(\cdot)),\ t \in [0, t_*]$, the state of the system (1) at a moment t, generated by control $u_{t-h}(\cdot) = (u(s),\ s \in [-h, t-h])$ and disturbance $w_t(\cdot) = (w(s),\ s \in [0, t])$.

Suppose also that a number $\delta_* > 0$ and a terminal system state $z_* \in R^n$ are given.

We are interested in a control

$$u(\cdot) = (u(t), t \in [-h, t_* - h]),$$

such that the following relations are satisfied

$$||z(t_*|u_{t_*-h}(\cdot), w_{t_*}(\cdot)) - z_*||^2_2 \leq \delta^2_* \ \text{ for all } w_{t_*}(\cdot) \in \Omega. \tag{3}$$

A control $u(\cdot) = u_{t_*-h}(\cdot)$ is said to be feasible if relations (3) hold.

The quality of control is defined by the cost functional

$$\int_0^{t_*-h} u^2(t)dt. \tag{4}$$

Then, the problem may be formulated as follows.

Open-loop worst-case formulation: find a control $u(\cdot)$, which minimizes the cost functional (4), and for which trajectories of the system (1) satisfy the relations (3).

This formulation belongs to the second type of problems mentioned in the Introduction. In such formulation, it happens quite often that no feasible control $u(\cdot)$ exists. Roughly speaking, in order to ensure feasibility, the set of admissible disturbances Ω should be a "small" neighborhood of the zero disturbance, and the parameter δ_* should be "large".

Let us change the problem formulation by including feedback aspects. Suppose that time moments

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = t_*,\ h < t_{i+1} - t_i,\ i = 0, ..., m \tag{5}$$

are given. For $i = 0, ..., m$, we will suppose that

a) at each (current) time moment t_i we will know the state

$$z(t_i) := z(t_i | u_{t_i - h}(\cdot), w_{t_i}(\cdot)) \tag{6}$$

of the real system (1) at the moment t_i and we will know the control $v_{i-1} = (u(t),\ t \in [t_i - h, t_i])$ that will be applied to the real system (1) during the time interval $t \in [t_i, t_i + h]$;

b) using this available information, at the moment t_i we may correct the future control $u(t), t \in [t_i, t_{i+1}]$, that will be applied to the real system during the time interval $[t_i + h, t_{i+1} + h]$, i.e. control $u(t), t \in [t_i, t_{i+1}]$, is a function of the known state $z(t_i)$ of the real system and a known control v_{i-1} constructed at the previous moment t_{i-1}.

Such a situation may take place in the following case (computation delay or controller-actuator delay). Suppose that at the moment t_i the controller knows the current state $z_i = z(t_i)$ of the real system (1). Using this information the controller starts to construct a new (or to correct old) control function that will be used in the real system in the future. This construction takes time and the new control function will be ready in h units of time. This means that the constructed control may be used in the real system only with delay h.

Consider control policy

$$\pi = (u_i(\cdot | z_i, v_{i-1}), i = 0, ..., m), \tag{7}$$

consisting of control laws

$$\begin{aligned} &u_i(\cdot | z_i, v_{i-1}) = (u_i(t | z_i, v_{i-1}), t \in T_i),\ i = 0, ..., m, \\ &z_i \in R^n,\ v_{i-1} = (u(t), t \in \Delta T_i),\ \Delta T_i = [t_i - h, t_i], \end{aligned} \tag{8}$$

at each control interval $T_i = [t_i, t_{i+1}]$, $i = 0, ..., m-1, T_m = [t_m, t_* - h]$.

Let us denote by $z(t|\pi, w(\cdot)) = \hat{z}(t)$, $t \in [0, t_*]$, the trajectory of the system

$$\begin{aligned} &\dot{\hat{z}}(t) = A\hat{z}(t) + bv^*(t-h) + gw(t),\ t \in [0, h], \\ &\dot{\hat{z}}(t) = A\hat{z}(t) + bu_i(t-h|\hat{z}(t_i), \hat{v}_{i-1}) + gw(t), \\ &t \in \tilde{T}_i,\ i = 0, ..., m,\ z(0) = z_0, \end{aligned} \tag{9}$$

with $\tilde{T}_i = [t_i + h, t_{i+1} + h[,\ i = 0, ..., m-1,\ \tilde{T}_m = [t_m + h, t_*]$,

$$\begin{aligned} &\hat{v}_{i-1} = (u_{i-1}(t|\hat{z}(t_{i-1}), \hat{v}_{i-2}(\cdot)), t \in \Delta T_i),\ i = 1, \ldots, m; \\ &\hat{v}_{-1} = (v^*(t), t \in \Delta T_0 = [-h, 0]). \end{aligned}$$

Now we may give a new mathematical formulation of the problem under investigation.

Closed-loop worst-case formulation: construct a control policy (7) consisting of control laws (8) for each control interval T_i, $i = 0, ..., m$, such that

- the corresponding trajectory $z(t|\pi, w(\cdot))$, $t \in [0, t_*]$, of the system (9) satisfies conditions

$$\|z(t_*|\pi, w(\cdot)) - z_*\|_2^2 \le \delta_*^2 \text{ for } \forall w(\cdot) \in \Omega; \tag{10}$$

- the guaranteed value of the cost functional

$$J(\pi) = \max_{w(\cdot) \in \Omega} \sum_{i=0}^{m} \int_{T_i} u_i^2(t|z(t_i|\pi, w(\cdot)), \hat{v}_{i-1}) dt \tag{11}$$

  takes the minimal value

$$\min_{\pi} J(\pi). \tag{12}$$

Thus, instead of one control function in the open-loop problem formulation, now we use a control policy (or strategy) that takes into account the possible corrections in the future, basing on available information about real system behavior.

Control policy (7) that solves the problem is called optimal and is denoted by

$$\begin{aligned} &\pi^0 = (u_i^0(\cdot|z_i, v_{i-1}),\ i = 0, ..., m), \\ &u_i^0(\cdot|z_i, v_{i-1}) = (u_i^0(t|z_i, v_{i-1}), t \in T_i), \\ &z_i \in R^n,\ v_{i-1} = (u(t), t \in \Delta T_i),\ i = 0, ..., m. \end{aligned} \tag{13}$$

This problem formulation belongs to the third type of problems mentioned in the Introduction. Solution of such problems is not a single fixed optimal control $u^0(t)$, $t \in [-h, t_* - h]$, but a control policy π^0, consisting of control laws. For each $i = 0, ..., m$, the concrete value of law $u_i^0(\cdot|z_i, v_{i-1})$ depends on the current state $z(t_i) = z_i$ of the system at the moments $t = t_i$ and the known control $v_{i-1} = (u(t), t \in \Delta T_i)$, constructed at the previous time moment t_{i-1}, that will be applied to the real system for the nearest time interval $t \in [t_i, t_i + h]$. Hence, concrete value of the law depends on the realized disturbance.

Note that the closed system (9) is a system with delay since (1) control $u_i^0(\cdot|z_i, v_{i-1})$ constructed at the moment t_i will be applied to the system with delay h, 2) the control $u_i^0(\cdot|z_i, v_{i-1})$ is determined on the basis of a known control $v_{i-1} = (u(t), t \in \Delta T_i)$ that was constructed at the previous time moment t_{i-1}.

Let us show that the class of feasible (guaranteed) controls can be essentially extended if we use a control policy (7) instead of one control function $u(t), t \in [-h, t_* - h]$.

First of all, let us introduce a class of admissible disturbances Ω. We define the set Ω as follows

$$\Omega := \{w(\cdot) : \int_{t_i}^{t_{i+1}} w^2(t)dt \leq \sigma_i, i = 0, \ldots, m\} \tag{14}$$

with given numbers $\sigma_i > 0$, $i = 0, ..., m$. This choice of ellipsoidal uncertainty is motivated by the fact that such bounds provide a good representation for uncertainties arising in many real control problems (see Matveev and Yakubovich, 1998; Savkin, Skafidas and Evans, 1999). Moreover, the class of bounded disturbances $|w(t)| \leq \alpha, t \in [0, t_*]$, belongs to the class of admissible disturbances (14) with the following choice of the numbers $\sigma_i = \alpha^2(t_{i+1} - t_i),\ i = 0, ..., m$.

It is evident that for the open-loop problem formulation, there exists an admissible control if and only if

$$\delta_*^2 \geq \gamma := \min_{u(\cdot)} \max_{w(\cdot) \in \Omega} ||z(t_*|u(\cdot), w(\cdot)) - z_*||_2^2. \tag{15}$$

Denote

$$Q_i = \int_{t_i}^{t_{i+1}} F(t_{i+1}, t)g(F(t_{i+1}, t)g)^T dt,\ i = 0, ..., m, \tag{16}$$

where $F(t, \tau)$ is the fundamental solution matrix of the system $\dot{x} = Ax$. Due to (2) we have $\det Q_i \neq 0$. Following Kostina and Kostyukova (2006), we can show that the relation

$$\gamma \geq \sum_{i=0}^{m} \sigma_i \lambda_{\max}(Q_i) \tag{17}$$

takes place. Moreover, based on results from Kostina and Kostyukova (2006) one can show that the relation

$$\delta_*^2 \geq \sigma_m \lambda_{\max}(Q_m) \tag{18}$$

is necessary and sufficient for the existence of a feasible policy π (see (7)) satisfying (10). Considering (15) and (17) one can see that the relation (18) is significantly weaker than the relation $\gamma \leq \delta_*^2$ that guarantees the existence of a feasible control in the open-loop worst-case problem formulation.

In what follows, we assume that the parameter $\delta_* > 0$ takes the minimal possible value, i.e. the equality always holds in (18):

$$\delta_*^2 = \sigma_m \lambda_{\max}(Q_m). \tag{19}$$

3. Optimal control policy π^0

In what follows, for an interval $t \in [t_i, t_{i+1}]$ we will distinguish two systems:

the real (or actual) system subject to a disturbance

$$\mathrm{RS}(i): \ \dot{z}(t) = \begin{cases} Az(t) + bv_{i-1}(t-h) + gw(t), t \in [t_i, t_i + h[, \\ Az(t) + br_i(t-h) + gw(t), t \in [t_i + h, t_{i+1}], \end{cases} \ z(t_i) = z_i,$$

and the nominal system (without disturbance)

$$\mathrm{NS}(i): \ \dot{x}(t) = \begin{cases} Ax(t) + bv_{i-1}(t-h), t \in [t_i, t_i + h[, \\ Ax(t) + br_i(t-h), t \in [t_i + h, t_{i+1}], \end{cases} \ x(t_i) = z_i.$$

As a rule, here, control

$$v_{i-1} := (v_{i-1}(t), t \in [t_i - h, t_i]) \tag{20}$$

is considered to be known, while control

$$r_i := (r_i(t), \ t \in [t_i, t_{i+1} - h])$$

should be determined.

Suppose that a control function r_i is chosen and denote by $x(t_{i+1}|z_i, v_{i-1}, r_i)$ the corresponding state of system NS(i) at the moment t_{i+1}. Let us denote by $Z_{i+1} \subset R^n$ the set of all system RS(i) states at the moment t_{i+1} that are generated by the same controls v_{i-1}, r_i and all admissible disturbances $w(t), t \in [t_i, t_{i+1}]$. It is evident that the set Z_{i+1} can be presented in the form

$$Z_{i+1} = \{z \in R^n : \tag{21}$$
$$z = x(t_{i+1}|z_i, v_{i-1}, r_i) + \int_{t_i}^{t_{i+1}} F(t_{i+1}, t) g w(t) dt, \int_{t_i}^{t_{i+1}} w^2(t) dt \leq \sigma_i\}.$$

The set Z_{i+1} is an important characteristic of the real system RS(i) since it is the only available information about the system RS(i) states at the moment t_{i+1} under assumption that we know the initial system state $z(t_i) = z_i$ and controls v_{i-1}, r_i. In our subsequent investigations, the set Z_{i+1} will play the essential role, that is why we are interested in a more constructive than (21) description of this set. The following Lemma gives us such a description.

LEMMA 1 *The set (21) can be presented in the form*

$$Z_{i+1}=Z_{i+1}(x(t_{i+1}|z_i,v_{i-1},r_i)) \tag{22}$$

with $Z_{i+1}(x) = \{z \in R^n : \|z - x\|^2_{Q_i^{-1}} \le \sigma_i\}$.

Here Q_i is defined in (16). The Lemma is proved in Kostina and Kostyukova (2006).

To derive the worst-case optimal policy π^0 (13) we apply Bellman's principle of optimality (see Bellman, 1961) and use dynamic programming and Lemma 1.

Suppose that we are at the real time moment t_m. By assumption, we know a real system RS(m) state $z_m := z(t_m)$ at this moment t_m and we know a control $v_{m-1} = (v_{m-1}(t),\ t \in [t_m - h, t_m])$ that will be applied to the real system during the time interval $t \in [t_m, t_m + h]$. Our aim is to find a suitable control $u_m = (u_m(t), t \in [t_m, t_{m+1} - h])$ that will be applied to the real system RS(m) over the interval $[t_m + h, t_{m+1}]$. Note that the control u_m has to ensure the fulfillment of the terminal conditions (10), i.e. the conditions

$$\|z(t_*) - z_*\|_2^2 \le \delta_*^2 \tag{23}$$

for all real system RS(m) states $z(t_*) = z(t_{m+1})$, generated by controls v_{m-1}, $r_m = u_m$ and all admissible disturbances $w(t), t \in [t_m, t_{m+1}]$. Taking into account presentation (22) we conclude that this condition is equivalent to the following one

$$\bar{\delta} := \max_{z} \|z - z_*\|_2^2 \quad \text{s.t. } z\in Z_{m+1}(x(t_{m+1}|z_m,v_{m-1},u_m)) \qquad \le \delta_*^2. \tag{24}$$

We introduce notation $\bar{z} = z - x(t_{m+1}|z_m, v_{m-1}, u_m)$ and $\bar{d} = x(t_{m+1}|z_m, v_{m-1}, u_m) - z_*$. Then we have

$$\bar{\delta} = \max_{\bar{z}} \|\bar{z} + \bar{d}\|_2^2 \ \text{s.t. } \bar{z}^T Q_m^{-1}\bar{z}\le\sigma_m \quad \ge \quad \max_{\bar{z}} \bar{z}^T\bar{z} \ \text{s.t. } \bar{z}^T Q_m^{-1}\bar{z}\le\sigma_m \quad = \quad \sigma_m\lambda_{\max}(Q_m).$$

Taking into account these relations, and (24), and (19), we conclude that terminal conditions will be fulfilled if and only if $\bar{d} = 0$. Thus, we have shown that a control u_m ensures the fulfillment of the terminal conditions (10) if and only if the corresponding state $x(t_{m+1}|z_m, v_{m-1}, u_m)$ of the nominal system NS(m) satisfies the condition $x(t_{m+1}|z_m, v_{m-1}, u_m) = z_*$.

Taking into account the cost functional (4), we come to the conclusion that, given a real system RS(m) state $z(t_m) = z_m$ and a control v_{m-1} that will be applied to the real system during the time interval $t \in [t_m, t_m + h]$, the optimal control law for the interval $[t_m, t_{m+1} - h]$ should solve the following optimal control problem

$$\min_{u_m} \int_{t_m}^{t_{m+1}-h} u_m^2(t)dt$$
$$\text{s.t. } \dot{x}(t) = \begin{cases} Ax(t) + bv_{m-1}(t-h), t \in [t_m, t_m + h[, \\ Ax(t) + bu_m(t-h), t \in [t_m + h, t_{m+1}], \end{cases}$$
$$x(t_m) = z_m, \; x(t_{m+1}) = z_*.$$

Consequently, basing on general control theory (see Pontryagin, Boltyanskij, Gamkrelidze and Mishchenko, 1986), we get that the optimal control law for the interval $[t_m, t_{m+1} - h]$, should be the following

$$u_m^0(t|z_m, v_{m-1}) = \psi_m^{0T}(z_m, v_{m-1})F(t_{m+1} - h, t)b, \; t \in [t_m, t_{m+1} - h],$$
$$\psi_m^0(z_m, v_{m-1}) = G_{m+1}^{-1}(z_* - F_{m+1}x_m(z_m, v_{m-1})). \tag{25}$$

Here and in what follows we will use the notation

$$x_i(z_i, v_{i-1}) = Fz_i + \int_{t_i}^{t_i+h} F(t_i + h, t)bv_{i-1}(t-h)dt, \; i = m, ..., 0,$$
$$F_i = F(t_i, t_{i-1} + h), \; G_i = \int_{t_{i-1}+h}^{t_i} F(t_i, t)b(F(t_i, t)b)^T dt, \; i = m+1, ..., 1,$$
$$F = F(h, 0), \; G = \int_0^h F(h, t)b(F(h, t)b)^T dt. \tag{26}$$

The optimal value of the cost functional for the time interval $[t_m, t_{m+1} - h]$ is equal to

$$J_m(z_m, v_{m-1}) = \psi_m^{0T}(z_m, v_{m-1})G_{m+1}\psi_m^0(z_m, v_{m-1}). \tag{27}$$

Now suppose that we are at the real time moment t_{m-1}. By assumption, at this moment we know a system state $z_{m-1} := z(t_{m-1})$ and we know a control v_{m-2} (see (20)) that will be applied to the real system RS$(m-1)$ during the time interval $t \in [t_{m-1}, t_{m-1} + h]$. Due to this information, we know the vector $x_{m-1}(z_{m-1}, v_{m-2})$ defined in (26).

Consider a control $u_{m-1}(t)$ $t \in [t_{m-1}, t_m]$ that will be applied to the real system during the time interval $t \in [t_{m-1} + h, t_m + h]$ and divide it into two parts

$$r_{m-1} = (r_{m-1}(t) := u_{m-1}(t), \; t \in [t_{m-1}, t_m - h]), \tag{28}$$
$$v_{m-1} = (v_{m-1}(t) := u_{m-1}(t), \; t \in [t_m - h, t_m]).$$

According to Lemma 1, the corresponding real system RS$(m-1)$ state $z_m := z(t_m)$ at time moment t_m satisfies the relation $z_m \in$

$Z_m(x(t_m|z_{m-1}, v_{m-2}, r_{m-1}))$. Consequently, the optimal guaranteed value of the cost functional for the time interval $[t_{m-1}, t_{m+1} - h]$ is equal to

$$\begin{aligned} & J_{m-1}(z_{m-1}, v_{m-2}) \\ & := \min_{r_{m-1}} \min_{v_{m-1}} \max_{z_m \in R^n} \Big(\int_{t_{m-1}}^{t_m - h} r_{m-1}^2(t)dt + \int_{t_m - h}^{t_m} v_{m-1}^2(t)dt + J_m(z_m, v_{m-1}) \Big) \\ & \quad \text{s.t. } z_m \in Z_m(x(t_m|z_{m-1}, v_{m-2}, r_{m-1})). \end{aligned} \tag{29}$$

We can show that in problem (29) one has to choose optimal controls r_{m-1}, v_{m-1} (see (28)) in the forms

$$\begin{aligned} & r_{m-1}(\varphi_{m-1}) = (r_{m-1}(t) = \varphi_{m-1}^T F(t_m - h, t)b,\ t \in [t_{m-1}, t_m - h]) , \\ & v_{m-1}(\psi_{m-1}) = (v_{m-1}(t) = \psi_{m-1}^T F(t_m, t)b,\ t \in [t_m - h, t_m]), \end{aligned} \tag{30}$$

with $\varphi_{m-1} \in R^n$, $\psi_{m-1} \in R^n$. Consequently, problem (29) can be written in the equivalent form

$$\begin{aligned} J_{m-1}(z_{m-1}, v_{m-2}) = \min_{\varphi_{m-1} \in R^n} \min_{\psi_{m-1} \in R^n} \max_{z_m \in R^n} \Big(& \varphi_{m-1}^T G_m \varphi_{m-1} \\ & + \psi_{m-1}^T G \psi_{m-1} + J_m(z_m, v_{m-1}(\psi_{m-1})) \Big) \\ \text{s.t. } \| z_m - F_m x_{m-1}(z_{m-1}, v_{m-2}) - G_m & \varphi_{m-1} \|^2_{Q^{-1}_{m-1}} \le \sigma_{m-1}. \end{aligned} \tag{31}$$

Here, notations (16) and (26) are used.

Let

$$\varphi_{m-1}^0(z_{m-1}, v_{m-2}), \quad \psi_{m-1}^0(z_{m-1}, v_{m-2})$$

be a solution to problem (31). Then, it follows from (28) and (30) that the optimal control law $u_{m-1}^0(\cdot|z_{m-1}, v_{m-2})$ from the optimal control policy is the following

$$\begin{aligned} & u_{m-1}^0(t|z_{m-1}, v_{m-2}) = \varphi_{m-1}^{0T}(z_{m-1}, v_{m-2}) F(t_m - h, t)b,\ t \in [t_{m-1}, t_m - h], \\ & u_{m-1}^0(t|z_{m-1}, v_{m-2}) = \psi_{m-1}^{0T}(z_{m-1}, v_{m-2}) F(t_m, t)b,\ t \in [t_m - h, t_m]. \end{aligned}$$

Suppose that we are at a real time moment t_i, $i \le m-1$, and function $J_{i+1}(z_{i+1}, v_i)$ has been determined with a vector $z_{i+1} \in R^n$ and a function $v_i = (v_i(t), t \in [t_{i+1} - h, t_{i+1}])$. Repeating the above arguments, we conclude that the optimal guaranteed value of the cost functional for the time interval $[t_i, t_{m+1} - h]$ is equal to

$$\begin{aligned} & J_i(z_i, v_{i-1}) \\ & := \min_{\varphi_i \in R^n, \psi_i \in R^n} \max_{z_{i+1} \in R^n} \Big(\varphi_i^T G_{i+1} \varphi_i + \psi_i^T G \psi_i + J_{i+1}(z_{i+1}, v_i(\psi_i)) \Big) \\ & \quad \text{s.t. } \| z_{i+1} - F_{i+1} x_i(z_i, v_{i-1}) - G_{i+1} \varphi_i \|^2_{Q_i^{-1}} \le \sigma_i. \end{aligned} \tag{32}$$

Here, $x_i(z_i, v_{i-1})$ is defined through (26),

$$v_i(\psi) = (\psi^T F(t_{i+1}, t)b,\ t \in [t_{i+1} - h, t_{i+1}]),\ \psi \in R^n. \tag{33}$$

Let

$$\varphi_i^0(z_i, v_{i-1}),\ \psi_i^0(z_i, v_{i-1}) \tag{34}$$

be a solution to problem (32). Then, the optimal control law $u_i^0(\cdot|z_i, v_{i-1})$ from the optimal control policy is the following

$$\begin{aligned} u_i^0(t|z_i, v_{i-1}) &= \varphi_i^{0T}(z_i, v_{i-1})F(t_{i+1} - h, t)b,\ t \in [t_i, t_{i+1} - h], \\ u_i^0(t|z_i, v_{i-1}) &= \psi_i^{0T}(z_i, v_{i-1})F(t_{i+1}, t)b,\ t \in [t_{i+1} - h, t_{i+1}]. \end{aligned} \tag{35}$$

At the initial moment $t_0 = 0$ we have

$$\begin{aligned} J_0(z_0, v_{-1}) := \min_{\varphi_0 \in R^n, \psi_0 \in R^n} \max_{z_1 \in R^n} &\Big(\varphi_0^T G_1 \varphi_0 + \psi_0^T G \psi_0 + J_1(z_1, v_0(\psi_0))\Big) \\ \text{s.t. } &\|z_1 - F_1 x_0(z_0, v_{-1}) - G_1 \varphi_0\|^2_{Q_0^{-1}} \le \sigma_0, \end{aligned} \tag{36}$$

with the known vector and function

$$\begin{aligned} x_0(z_0, v_{-1}) &= F z_0 + \int_{t_0}^{t_0+h} F(t_0 + h, t) b v^*(t - h) dt, \\ v_{-1} &= (v_{-1}(t) := v^*(t), t \in [-h, 0]). \end{aligned} \tag{37}$$

Remark 1 *The function $J_i(z_i, v_{i-1})$ and relations (32) may be interpreted as Bellman's function and Bellman's equation, respectively. Then, the control defined in (34), (35) is the solution of the Bellman's equation.*

Remark 2 *Note that problem (32) can be represented in the form*

$$\begin{aligned} &J_i(z_i, v_{i-1}) \\ = &\min_{\varphi_i \in R^n, \psi_i \in R^n} \max_{z_{i+1} \in R^n} \min_{\varphi_{i+1} \in R^n, \psi_{i+1} \in R^n} \max_{z_{i+2} \in R^n} \dots \min_{\varphi_{m-1} \in R^n, \psi_{m-1} \in R^n} \max_{z_m \in R^n} \\ &\Big(\sum_{s=i}^{m-1} (\varphi_s^T G_{s+1} \varphi_s + \psi_s^T G \psi_s) + \|z_* - F_{m+1}(F z_m + G \psi_{m-1})\|^2_{G_{m+1}^{-1}}\Big) \\ &\text{s.t. } \|z_s - F_s(F z_{s-1} + G \psi_{s-2}) - G_s \varphi_{s-1}\|^2_{Q_{s-1}^{-1}} \le \sigma_{s-1},\ s = i+2, ..., m, \\ &\qquad \|z_{i+1} - F_{i+1} x_i(z_i, v_{i-1}) - G_{i+1} \varphi_i\|^2_{Q_i^{-1}} \le \sigma_i. \end{aligned}$$

Problem (36) involves a sequence of min-max optimizations, formulated as dynamic programs. Since the min-max problem at each stage does not yield a simple analytical solution, it must be solved numerically. The numerical procedure involves discretizing the states at each stage and computing and storing the min-max costs $J_i(z_i, v_{i-1})$ (32) and the corresponding vectors $\varphi_i^0(z_i, v_{i-1})$, $\psi_i^0(z_i, v_{i-1})$ (see (34)) for all combinations of the discretized states $x_i(z_i, v_{i-1}) \in R^n$ and for all stages $i = 1, ..., m$. Undoubtedly, the above procedure is numerically demanding and suffers from the curse of dimensionality. As the dynamic programming solution is computationally prohibitive, it is of practical interest to develop a suboptimal, but more computationally amenable algorithms, which can potentially be implemented on-line. In the next Section, we consider some approximations of the problems (32) that lead to a practical algorithm for construction of an approximative policy.

4. Approximative policy $\tilde{\pi}^0$

4.1. Auxiliary results and notation

Let $a \in R^n, b \in R^n$ be given vectors, F_*, F, D, G_*, G, $Q \in R^{n\times n}$ be given matrices, matrices F_*, F be nonsingular and matrices D, G_*, G, Q be positive definite, $\sigma > 0$ be a given number. Consider two problems

$$I^0 := \min_{\varphi\in R^n, \psi\in R^n} \max_{z\in R^n} \Big(\varphi^T G_* \varphi + \psi^T G \psi + \|a - F_*(Fz + G\psi)\|_D^2\Big) \quad (38)$$
$$\text{s.t. } \|z - b - G_*\varphi\|^2_{Q^{-1}} \le \sigma,$$

and

$$I^* := \min_{\lambda\in R, \lambda\ge\mu} (d^T D(\lambda) d + \lambda\sigma) \quad (39)$$

where $\quad \mu = \lambda_{\max}(M^{-T}\bar{D}M^{-1}),\ Q^{-1} = M^T M,\ \ d = b - (F_*F)^{-1}a,$

$$D(\lambda) = \Big(\bar{D}^{-1} + G_* + F^{-1}GF^{-T} - \frac{Q}{\lambda}\Big)^{-1},\ \bar{D} = (F_*F)^T D F_* F. \quad (40)$$

Lemma 2 *The relationship $I^0 = I^*$ is true. Let λ^0 be an optimal solution to problem (39), then a solution to problem (38) is the following*

$$\psi^0 = -F^{-T}D(\lambda^0)(b - (F_*F)^{-1}a),\ \varphi^0 = -D(\lambda^0)(b - (F_*F)^{-1}a). \quad (41)$$

This lemma is proved in the Appendix.

Let us introduce the notations that will be needed in this section:

$$a_{m+1} = z_*,\ a_i = (F_{i+1}F)^{-1}a_{i+1},\ i = m, ..., 1,$$
$$d_i(z_{i-1}, v_{i-2}) = F_i x_i(z_{i-1}, v_{i-2}) - a_i,\ i = m+1, ..., 1; \quad (42)$$

$$D_{m+1} = G_{m+1}^{-1},\ \bar{D}_{m+1} = F^T F_{m+1}^T D_{m+1} F_{m+1} F,$$
$$D_m(\lambda_m) = \Big(\bar{D}_{m+1}^{-1} + G_m + F^{-1}GF^{-T} - \frac{Q_{m-1}}{\lambda_m}\Big)^{-1}, \quad (43)$$

$$\mu_m = \lambda_{\max}(M_{m-1}^{-T}\bar{D}_{m+1}M_{m-1}^{-1}),\ Q_{m-1}^{-1} = M_{m-1}^T M_{m-1},$$
$$\rho_m(z_m, v_{m-1}) = d_{m+1}^T(z_m, v_{m-1}) D_{m+1} d_{m+1}(z_m, v_{m-1}), \quad (44)$$

$$
\begin{aligned}
&D_i(\lambda_i, ..., \lambda_m) = \Big(\bar{D}^{-1}_{i+1}(\lambda_{i+1}, ..., \lambda_m) + G_i + F^{-1}GF^{-T} - \frac{Q_{i-1}}{\lambda_i}\Big)^{-1}, \\
&\bar{D}_{i+1}(\lambda_{i+1}, ..., \lambda_m) = F^T F^T_{i+1} D_{i+1}(\lambda_{i+1}, ..., \lambda_m) F_{i+1} F, && (45) \\
&\mu_i = \mu_i(\lambda_{i+1}, ..., \lambda_m) = \lambda_{\max}(M^{-T}_{i-1}\bar{D}_{i+1}(\lambda_{i+1}, ..., \lambda_m) M^{-1}_{i-1}), \\
&\rho_i(z_i, v_{i-1}, \lambda_{i+1}, ..., \lambda_m) && (46) \\
&= d^T_{i+1}(z_i, v_{i-1}) D_{i+1}(\lambda_{i+1}, ..., \lambda_m) d_{i+1}(z_i, v_{i-1}) + \sum_{s=i+1}^{m} \lambda_s \sigma_{s-1}, \\
&Q^{-1}_i = M^T_i M_i,\ i = m-1, ..., 0.
\end{aligned}
$$

4.2. Justification of approximative policies

According to the results from Section 3, the optimal control laws forming the optimal control policy π^0 are constructed by the rule (25) and by the rules (35) using the solutions (34) of problems (32) for $i = m-1, \dots, 0$.

Denote $I_m(z_m, v_{m-1}) := J_m(z_m, v_{m-1})$. Let us investigate the problems (32) for $i = m-1, \dots, 0$.

We put $i = m-1$ and consider the corresponding problem (32) that can be written in the form

$$
\begin{aligned}
&J_{m-1}(z_{m-1}, v_{m-2}) = I_{m-1}(z_{m-1}, v_{m-2}) \\
:= &\min_{\varphi_{m-1}, \psi_{m-1}} \max_{z_m} \Big(\varphi^T_{m-1} G_m \varphi_{m-1} + \psi^T_{m-1} G \psi_{m-1} + \rho_m(z_m, v_{m-1}(\psi_{m-1}))\Big) \\
&\text{s.t. } \|z_m - F_m x_{m-1}(z_{m-1}, v_{m-2}) - G_m \varphi_{m-1}\|^2_{Q^{-1}_{m-1}} \le \sigma_{m-1}.
\end{aligned}
$$

According to Lemma 2 we have

$$
I_{m-1}(z_{m-1}, v_{m-2}) = \min_{\lambda_m \ge \mu_m} \rho_{m-1}(z_{m-1}, v_{m-2}, \lambda_m). \tag{47}
$$

Here, functions $\rho_m(z_m, v_{m-1})$, $\rho_{m-1}(z_{m-1}, v_{m-2}, \lambda_m)$ and $v_{m-1}(\psi)$ are defined by (44), (46) and (33). Let $\lambda^0_m = \lambda^0_m(z_{m-1}, v_{m-2})$ be a solution to problem (47). Then the vectors

$$
\begin{aligned}
&\varphi^0_{m-1}(z_{m-1}, v_{m-2}) = -D_m(\lambda^0_m) d_m(z_{m-1}, v_{m-2}), \\
&\psi^0_{m-1}(z_{m-1}, v_{m-2}) = -F^{-T} D_m(\lambda^0_m) d_m(z_{m-1}, v_{m-2}), && (48)
\end{aligned}
$$

form the solution to problem (32) (with $i = m-1$) and the optimal control law $u^0_{m-1}(\cdot|z_{m-1}, v_{m-2})$ is constructed by the rules (35) (with $i = m-1$).

Now we put $i = m-2$ and consider the corresponding problem (32). Taking into account the relationship (47), we can rewrite the problem in the form

$$\begin{aligned} & J_{m-2}(z_{m-2}, v_{m-3}) \\ & = \min_{\varphi_{m-2},\psi_{m-2}} \max_{z_{m-1}} \Big(\varphi_{m-2}^T G_{m-1} \varphi_{m-2} + \psi_{m-2}^T G \psi_{m-2} \\ & \qquad + I_{m-1}(z_{m-1}, v_{m-1}(\psi_{m-2})) \Big) \\ & = \min_{\varphi_{m-2},\psi_{m-2}} \max_{z_{m-1}} \min_{\lambda_m \geq \mu_m} \Big(\varphi_{m-2}^T G_{m-1} \varphi_{m-2} + \psi_{m-2}^T G \psi_{m-2} + \\ & \qquad + \rho_{m-1}(z_{m-1}, v_{m-2}(\psi_{m-2}), \lambda_m) \Big) \\ & \text{ s.t. } \|z_{m-1} - F_{m-1} x_{m-2}(z_{m-2}, v_{m-3}) - G_{m-1}\varphi_{m-2}\|^2_{Q^{-1}_{m-2}} \leq \sigma_{m-2}. \end{aligned} \tag{49}$$

Using (49) and min-max inequality, we may conclude that

$$J_{m-2}(z_{m-2}, v_{m-3}) \leq I_{m-2}(z_{m-2}, v_{m-3}) \tag{50}$$

where

$$\begin{aligned} I_{m-2}(z_{m-2}, v_{m-3}) := & \min_{\lambda_m \geq \mu_m} \min_{\varphi_{m-2},\psi_{m-2}} \max_{z_{m-1}} \Big(\varphi_{m-2}^T G_{m-1} \varphi_{m-2} + \\ & + \psi_{m-2}^T G \psi_{m-2} + \rho_{m-1}(z_{m-1}, v_{m-2}(\psi_{m-2}), \lambda_m) \Big) \\ & \text{ s.t. } \|z_{m-1} - F_{m-1} x_{m-2}(z_{m-2}, v_{m-3}) - G_{m-1}\varphi_{m-2}\|^2_{Q^{-1}_{m-2}} \leq \sigma_{m-2}. \end{aligned} \tag{51}$$

According to Lemma 2 and (51) we have

$$I_{m-2}(z_{m-2}, v_{m-3}) = \min_{\lambda_m \geq \mu_m} \min_{\lambda_{m-1} \geq \mu_{m-1}(\lambda_m)} \rho_{m-2}(z_{m-2}, v_{m-3}, \lambda_{m-1}, \lambda_m), \tag{52}$$

where $\rho_{m-2}(z_{m-2}, v_{m-3}, \lambda_{m-1}, \lambda_m)$ is defined by (46). It follows from (50) that problem (51) binds the problem (49) (or problem (32) with $i = m-2$) from above.

Let $\lambda^0_{m-1} = \lambda^0_{m-1}(z_{m-2}, v_{m-3})$, $\lambda^0_m = \lambda^0_m(z_{m-2}, v_{m-3})$ be a solution to problem (52). Then, according to Lemma 2, the vectors

$$\begin{aligned} & \tilde{\varphi}_{m-2}(z_{m-2}, v_{m-3}) = -D_{m-1}(\lambda^0_{m-1}, \lambda^0_m) d_{m-1}(z_{m-2}, v_{m-3}), \\ & \tilde{\psi}_{m-2}(z_{m-2}, v_{m-3})) = -F^{-T} D_{m-1}(\lambda^0_{m-1}, \lambda^0_m) d_{m-1}(z_{m-2}, v_{m-3}) \end{aligned} \tag{53}$$

solve the problem (51).

We construct a control law $\tilde{u}^0_{m-2}(\cdot|z_{m-2}, v_{m-3})$ by the rules (35) with $i = m-2$ and the vectors (34) (that solve the problem (32)) replaced by the vectors (53) (that solve the approximate problem (51)). We can show that, for fixed z_{m-2}, v_{m-3}, using the constructed control laws $u^0_m(\cdot|z_m, v_{m-1})$,

$u^0_{m-1}(\cdot|z_{m-1}, v_{m-2})$, and $\tilde{u}^0_{m-2}(\cdot|z_{m-2},\, v_{m-3})$ for the interval $[t_{m-2}, t_{m+1}-h]$, we get guaranteed value of the cost functional equal to $I_{m-2}(z_{m-2}, v_{m-3})$.

Repeating these arguments and operations recursively, we get the following. For each $i \leq m-2$, we have

$$J_i(z_i, v_{i-1}) \leq I_i(z_i, v_{i-1}) \tag{54}$$

where

$$\begin{aligned} &I_i(z_i, v_{i-1}) \\ &:= \min_{\varphi_i \in R^n, \psi_i \in R^n} \max_{z_{i+1} \in R^n} \left(\varphi_i^T G_{i+1} \varphi_i + \psi_i^T G \psi_i + I_{i+1}(z_{i+1}, v_i(\psi_i)) \right) \\ &\qquad \text{s.t. } \|z_{i+1} - F_{i+1} x_i(z_i, v_{i-1}) - G_{i+1}\varphi_i\|^2_{Q_i^{-1}} \leq \sigma_i, \end{aligned} \tag{55}$$

$x_i(z_i, v_{i-1})$ is defined through (26). Due to Lemma 2 we get

$$I_i(z_i, v_{i-1}) = \min_{\substack{\lambda_s \geq \mu_s(\lambda_{s+1}, \ldots, \lambda_m), \\ s = i+1, \ldots, m}} \rho_i(z_i, v_{i-1}, \lambda_{i+1}, ..., \lambda_m), \tag{56}$$

functions $\mu_s(\lambda_{s+1} \ldots, \lambda_m)$, $\rho_i(z_i, v_{i-1}, \lambda_{i+1}, ..., \lambda_m)$ are given by relations (45), (46). It follows from (54) that problem (55) binds the problem (32) from above, and thus the optimal value of the cost function in (55) gives an upper bound of the cost function in (32).

Let $\lambda^0_{i+1} = \lambda^0_{i+1}(z_i, v_{i-1}), \ldots, \lambda^0_m = \lambda^0_m(z_i, v_{i-1})$ be a solution to problem (56). Then, according to Lemma 2, the vectors

$$\begin{aligned} \tilde{\varphi}_i(z_i, v_{i-1}) &= -D_{i+1}(\lambda^0_{i+1}, ..., \lambda^0_m) d_{i+1}(z_i, v_{i-1}), \\ \tilde{\psi}_i(z_i, v_{i-1}) &= -F^{-T} D_{i+1}(\lambda^0_{i+1}, ..., \lambda^0_m) d_{i+1}(z_i, v_{i-1}), \end{aligned} \tag{57}$$

solve the problem (55). Using these vectors, let us determine a control law $\tilde{u}^0_i(t|z_i, v_{i-1})$, $t \in [t_i, t_{i+1}]$, by the rules (35), where the vectors (34) (that solve the problem (32)) are replaced with the vectors (57) (that solve problem (55) approximating problem (32)):

$$\begin{aligned} \tilde{u}^0_i(t|z_i, v_{i-1}) &= \tilde{\varphi}_i^T(z_i, v_{i-1}) F(t_{i+1}-h, t) b,\ t \in [t_i, t_{i+1}-h], \\ \tilde{u}^0_i(t|z_i, v_{i-1}) &= \tilde{\psi}_i^T(z_i, v_{i-1}) F(t_{i+1}, t) b,\ t \in [t_{i+1}-h, t_{i+1}]. \end{aligned} \tag{58}$$

It can be shown that for each $0 \leq s \leq m$ and for fixed available data z_s, v_{s-1}, using the constructed control laws (57), (58) with $i = s, ..., m-1$ and the control law (25), we get guaranteed value of the cost functional at the interval $[t_s, t_{m+1}-h]$ equal to $I_s(z_s, v_{s-1})$.

Hence, we have constructed a control policy $\tilde{\pi}^0$ with the control laws $\tilde{u}^0_i(\cdot|z_i, v_{i-1})$ determined by the rules (57), (58) for $i = 0, \ldots, m-1$, and by the rule (25) for $i = m$. We have also shown that, for this policy, the guaranteed value of the cost functional is equal to I^0:

$$I^0 := I_0(z_0, v_{-1}) = \min_{\substack{\lambda_s \geq \mu_s(\lambda_{s+1}, \ldots, \lambda_m), \\ s = 1, \ldots, m}} \rho_0(z_0, v_{-1}, \lambda_1, \ldots, \lambda_m). \tag{59}$$

Here, $\rho_0(z_0, v_{-1}, \lambda_1, \ldots, \lambda_m)$ is defined in (46). Problem (59) can be rewritten in the form

$$I_0(x_0) \tag{60}$$
$$:= \min_{\lambda_1 \geq \mu_1(\lambda_2,\ldots,\lambda_m)} \cdots \min_{\lambda_{m-1} \geq \mu_{m-1}(\lambda_m)} \min_{\lambda_m \geq \mu_m} \Big(d_1^T D_1(\lambda_1, \ldots, \lambda_m) d_1 + \sum_{s=1}^{m} \lambda_s \sigma_{s-1} \Big),$$

where $d_1 = F_1 x_0 - a_1$ and $x_0 = x_0(z_0, v_{-1})$ is defined in (37). Note that the problem (60) coincides with problem (56) when $i = 0$.

Lemma 3 *For any $i = 0, \ldots, m-1$, problem (56) is a convex mathematical programming problem w.r.t. $m - i$ decision variables $\lambda_s, s = i+1, \ldots, m$.*

Proof. Lemma 3 follows from Lemmas 6 and 7, proved in the Appendix. ∎

For any $i = 0, \ldots, m-1$, problem (56) has a solution that can be easily found by standard convex optimization methods. Having a solution to problem (56), one can easily construct control law $\tilde{u}_i^0(\cdot|z_i, v_{i-1})$ by rules (57), (58) for each $i = 0, \ldots, m-1$.

Let us propose an approximative policy that gives the same guaranteed value of the cost functional as the policy $\tilde{\pi}^0$, but its construction needs less computational effort.

We solve convex mathematical programming problem (60) and find its solution

$$\lambda_1^0 = \lambda_1^0(z_0, v_{-1}), \ldots, \lambda_m^0 = \lambda_m^0(z_0, v_{-1}).$$

Using only this solution we determine control laws $\tilde{u}_i^*(\cdot|z_i, v_{i-1})$, $i = 0, \ldots, m-1$, by the rules (57) and (58) and law $u_m^0(\cdot|z_m, v_{m-1})$ by rules (25). We denote the policy with such control laws by $\tilde{\pi}^*$.

One can prove that the guaranteed value of the cost function for the policy $\tilde{\pi}^*$ is equal to $I^0 = I_0(z_0, v_{-1})$, just like for the policy $\tilde{\pi}^0$. But, in general, for a concrete admissible disturbance $w(\cdot)$, the following inequality will take place

$$J(\tilde{\pi}^0, w(\cdot)) := \sum_{i=0}^{m} \int_{T_i} \Big(\tilde{u}_i^0(t|z(t_i|\tilde{\pi}^0, w(\cdot)), \tilde{v}_{i-1}^0) \Big)^2 dt$$
$$\leq J(\tilde{\pi}^*, w(\cdot)) := \sum_{i=0}^{m} \int_{T_i} \Big(\tilde{u}_i^*(t|z(t_i|\tilde{\pi}^*, w(\cdot)), \tilde{v}_{i-1}^*) \Big)^2 dt.$$

Here

$$\tilde{v}^0_{i-1} = \left(\tilde{u}^0_{i-1}\left(t|z(t_{i-1}|\tilde{\pi}^0, w(\cdot)), \tilde{v}^0_{i-2}\right), t \in \Delta T_i\right), \ i = 1, \dots, m;$$
$$\tilde{v}^*_{i-1} = \left(\tilde{u}^*_{i-1}\left(t|z(t_{i-1}|\tilde{\pi}^*, w(\cdot)), \tilde{v}^*_{i-2}\right), t \in \Delta T_i\right), \ i = 1, \dots, m;$$
$$\tilde{v}^0_{-1} = \tilde{v}^*_{-1} = (v^*(t), t \in \Delta T_0 = [-h, 0]).$$

Let us summarize the results of this subsection.

We justified and constructed the control policies $\tilde{\pi}^0$ and $\tilde{\pi}^*$ that **for any admissible disturbance** $w(\cdot) \in \Omega$ guarantee

• that the initial state z_0 of the actual system (1) is steered into the δ_*-neighborhood of the terminal state z_* in m steps;

• the value of the cost functional at the realized control does not exceed I^0;

• this estimation is exact:

$$I^0 = \max_{w(\cdot)\in\Omega} J(\tilde{\pi}^0, w(\cdot)) = \max_{w(\cdot)\in\Omega} J(\tilde{\pi}^*, w(\cdot)); \tag{61}$$

• the presented policies $\tilde{\pi}^0$ and $\tilde{\pi}^*$ can be easily constructed.

Let us denote by $w_{\tilde{\pi}^0}(\cdot) = w_{\tilde{\pi}^*}(\cdot)$ an admissible disturbance at which maximum in (61) is reached. In other words, $w_{\tilde{\pi}^0}(\cdot)$ is the worst-case disturbance for policies $\tilde{\pi}^0$ and $\tilde{\pi}^*$. Let us show how one can find this disturbance.

For $i = 0, ..., m$, knowing $z^0_i = z(t_i)$ and $\varphi^0_i := \tilde{\varphi}_i(z^0_i, v_{i-1})$, $\psi^0_i := \tilde{\psi}_i(z^0_i, v_{i-1})$, we can find the worst-case real system state $z^0_{i+1} = z(t_{i+1})$ as a solution to the problem

$$\max_{z_{i+1}} \|F_{i+2}(Fz_{i+1} + G\psi^0_i) - a_{i+2}\|^2_{D_{i+2}(\lambda^0_{i+2},\dots,\lambda^0_m)}$$
$$\text{s.t. } \|z_{i+1} - F_{i+1}(Fz^0_i + G\psi^0_{i-1}) - G_{i+1}\varphi^0_i\|^2_{Q^{-1}_i} \le \sigma_i.$$

For policies $\tilde{\pi}^0$ and $\tilde{\pi}^*$, the worst-case admissible disturbance $w_{\tilde{\pi}^0}(\cdot)$ can be constructed by the rules

$$w_{\tilde{\pi}^0}(t) = \xi^{0T}_i F(t_{i+1}, t)g, \ t \in [t_i, t_{i+1}], \tag{62}$$
$$\xi^0_i = Q^{-1}_i(z^0_{i+1} - F_{i+1}(Fz^0_i + G\psi^0_{i-1}) - G_{i+1}\varphi^0_i), \ i = 0, ..., m.$$

4.3. MPC interpretation

Let us show how the application of the described policy in real processes can be interpreted in Model Predictive Control (MPC) scheme in the shrinking horizon style. Such a scheme is used to control dynamic systems for a given finite time interval $[0, t_*]$. In such scheme, at a current time t_i a predictive problem is considered for the control interval $[t_i, t_*]$. The type of MPC is determined by the type of the predictive problem which is solved on-line. The type of the predictive problem depends on how a disturbance is taken into account (Bemporad, Borelli and Morari, 2003).

For the problem under consideration in this paper, following Bemporad, Borelli and Morari (2003), and Lee and Yu (1997), let us use the MPC in shrinking horizon style with a predictive problem of type III (see Introduction). As a result we obtain the following MPC strategy.

Let t_i be a current time moment and $z_i = z(t_i)$ be a known current state of the system (1) and v_{i-1} be a known control that will be applied to the real system during time interval $t \in [t_i, t_i + h]$. Using this information, we compute a control that will be applied to the real system for $[t_i + h, t_{i+1} + h]$ by the rules (35), where vectors (34) solve the problem $J_i(z_i, v_{i-1})$, defined in (32). In other words, one has to solve on-line the predictive problem $J_i(z_i, v_{i-1})$ and use the i-th law of the optimal policy π^0 with known state z_i and control v_{i-1}.

The problem $J_i(z_i, v_{i-1})$ is difficult to solve, especially on-line. Hence, we replace the predictive problem $J_i(z_i, v_{i-1})$ by an approximative problem $I_i(z_i, v_{i-1})$ (see (55)) and the control law (35) is replaced by the control law (58) that is constructed on the basis of solution to the problem $I_i(z_i, v_{i-1})$. The advantage of this replacement is that the problem $I_i(z_i, v_{i-1})$ is easy to solve on-line for given z_i and v_{i-1}. We suppose that these calculations take time not exceeding h units of time. Hence, the new control function (58) will be ready to be applied to the real system at the moment $t_i + h$.

5. Classical feedback

To illustrate the effectiveness of the proposed approximative control policy $\tilde{\pi}^0$ we compare it with another reasonably performing control policy. In this section, we briefly describe this control policy that is based on ideas of classical feedback. In the next section, we will compare the properties of these policies in numerical examples.

Suppose that we are at the current time moment t_i. By assumption we know the system state z_i at the time moment t_i and control v_{i-1}. Hence we know the state $x(t_i+h) = x_i(z_i, v_{i-1})$ (see (26)) of the nominal system NS(i) at $t = t_i+h$. We compute a control $u(t), t \in [t_i, t_* - h]$, that solves the problem

$$\min_u \int_{t_i}^{t_*-h} u^2(t)dt \tag{63}$$

$$\text{s.t. } \dot{x}(t) = Ax(t) + bu(t-h),\ x(t_i+h) = x_i(z_i, v_{i-1}),\ x(t_*) = z_*.$$

Control is given by

$$\bar{u}_i(t-h|z_i, v_{i-1}) = \psi_i^T(z_i, v_{i-1})F(t_*, t)b, t \in [t_i + h, t_*], \tag{64}$$

$$\psi_i^T(z_i, v_{i-1}) = (z_* - F(t_*, t_i + h)x_i(z_i, v_{i-1}))^T \bar{G}_i^{-1}, \tag{65}$$

$$\bar{G}_i = \int_{t_i+h}^{t_*} F(t_*, t)b(F(t_*, t)b)^T dt.$$

This control is applied to the actual system at the interval $[t_i + h, t_{i+1} + h]$, bringing the actual (perturbed) system to the state z_{i+1} at the moment t_{i+1} :

$$z_{i+1} = F(t_{i+1}, t_i + h)x_i(z_i, v_{i-1}) + G_{i+1}F^T(t_*, t_{i+1})\psi_i + \int_{t_i}^{t_{i+1}} F(t_{i+1}, t)gw(t)dt.$$

At the moment t_{i+1}, it is supposed that the real system state z_{i+1} and control $v_i = (\bar{u}_i(t), t \in [t_{i+1} - h, t_{i+1}])$ are available. Hence, by replacing i by $i + 1$ we can calculate a vector $x_{i+1}(z_{i+1}, v_i)$ by (26) and correct the control by the rule (64). This process may be continued. As a result we get a policy $\bar{\pi} = (\bar{u}_i(\cdot|z_i, v_{i-1}),\ i = 0, \ldots, m)$ of type (7) with the controls (8)

$$\bar{u}_i(\cdot|z_i, v_{i-1}) = (z_* - F(t_*, t_i + h)x_i(z_i, v_{i-1}))^T \bar{G}_i^{-1} F(t_*, t)b,\ t \in [t_i, t_{i+1}[.$$

The (guaranteed) value of the cost functional (11) of the policy $\bar{\pi}$ is equal to

$$J(\bar{\pi}) = \max_{w(\cdot)\in\Omega} J(\bar{\pi}, w(\cdot)),\ J(\bar{\pi}, w(\cdot)) := \sum_{i=0}^{m} \int_{T_i} \bar{u}_i^2(t|z(t_i|\bar{\pi}, w(\cdot)), v_{i-1})dt. \tag{66}$$

In the proposed policy $\bar{\pi}$ we are able to correct future control at the current moment t_i on the basis of information that is available at this moment. The information is the same as used for constructing policies $\tilde{\pi}^0$ and π^0. However, in policies $\tilde{\pi}^0$ and π^0, at the moment t_i we perform the correction taking into account that we will be able to correct the control in the future moments t_{i+1}, ..., t_m, on the basis of new information that will be available at these moments. In the policy $\bar{\pi}$, described in this section, based on the principles of classical feedback, one does not include the fact that more information about the states becomes available as time progresses.

Note that the policy $\bar{\pi}$ can be interpreted in MPC style as well. But in this case, we use another type of predictive problem, namely problem (63). When formulating this problem, one supposes that there is no disturbance in the future interval $[t_i, t_*]$.

6. Numerical experiments

In this section, we present results of numerical comparison of the policies suggested in this paper. In our numerical experiments, we consider the dynamic system (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{67}$$

and two sets of data: **A)** and **B)** as follows

A) $b = (0, -1, 0, 1)^T,\ g = (2, 1, 4, 1)^T,$
$z_0 = (1/3, 1/3, 1/3, 1/3)^T,\ z_* = (0, 0, 0, 0)^T,$

and

B) $b = (0, 1/2, 0, 1/2)^T,\ g = (1, 1, 1, 1)^T,$
$z_0 = -(5, 5, 5, 5)^T,\ z_* = (6, 6, 6, 6)^T.$

The policies $\tilde{\pi}^0$ and $\bar{\pi}$ were tested on several sets $T^s_m = \{t_0, \dots, t_m, t_{m+1} = t_*\}$, $s = 1, 2;\ m \in \{3, 5, 6, 8, 9, 10\}$, of correction points presented in Table 1 for the dynamic system (1) with data (67), A) and $h = 0.05$; and in Table 2 for the dynamic system (1) with data (67), B) and $h = 4$. Note that $T^s_k \subset T^s_p$ if $p > k$, $s = 1, 2$.

Table 1. Sets of correction points for dynamic system with data A

	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}
T^1_{10}	0	0.35	0.65	1.05	1.55	1.85	2.15	2.45	2.75	2.9	3.05	5
T^1_9	0	0.35	0.65	1.05	1.55	1.85	2.15	2.45	2.75	3.05	5	
T^1_8	0	0.35	0.65	1.05	1.55	1.85	2.15	2.45	3.05	5		
T^1_6	0	0.35	1.05	1.55	1.85	2.45	3.05	5				
T^1_5	0	0.35	1.05	1.55	2.45	3.05	5					
T^1_3	0	1.05	2.45	3.05	5							

Table 2. Sets of correction points for dynamic system with data B

	t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}
T^2_{10}	0	24	44	74	94	109	124	139	154	164	174	200
T^2_9	0	24	44	74	94	109	124	154	164	174	200	
T^2_8	0	24	44	74	94	124	154	164	174	200		
T^2_6	0	44	74	124	154	164	174	200				
T^2_5	0	74	124	154	164	174	200					
T^2_3	0	74	154	174	200							

For any set of correction points T^s_m , $s = 1, 2;\ \ m \in \{3, 5, 6, 8, 9, 10\}$, the corresponding set of admissible disturbances Ω is defined through (14), where

$$\sigma_i = \alpha^2(t_{i+1} - t_i),\ i = 0, \dots, m,$$
$$\text{with } \alpha = 1/4 \text{ in A), and } \alpha = 1/5 \text{ in B).} \tag{68}$$

Note that the class of bounded disturbances $|w(t)| \leq \alpha, t \in T$, belongs to the set of admissible disturbances.

Tables 3 and 4 contain the following information. The first rows of the tables give guaranteed values I^0 of the cost functional for the policy $\tilde{\pi}^0$ applied to the dynamic systems (1), (67), A) (see Table 3) and (1), (67), B) (see Table 4), respectively. According to the results of Section 4, for any set T_m^i of correction points, in order to calculate I^0 we have to solve the corresponding convex mathematical programming problem (60). In our experiments, this problem was successfully solved by means of embedded MATLAB-function.

Table 3. Values of I^0 and $J(\bar{\pi}, \bar{w}(\cdot))$ corresponding to correction sets from Table 1

set of correction points	T_3^1	T_5^1	T_6^1	T_8^1	T_9^1	T_{10}^1
I^0	672.934	597.147	552.052	522.961	274.211	264.203
$J(\bar{\pi}, \bar{w}(\cdot))$	865.448	781.242	731.126	690.065	504.512	468.655

Table 4. Values of I^0 and $J(\bar{\pi}, \bar{w}(\cdot))$ corresponding to correction sets from Table 2

set of correction points	T_3^2	T_5^2	T_6^2	T_8^2	T_9^2	T_{10}^2
I^0	699.51	174.465	142.6317	102.709	97.505	67.005
$J(\bar{\pi}, \bar{w}(\cdot))$	789.54	270.786	233.402	186.875	174.852	149.087

The second rows of Tables 3 and 4 contain the values of some estimates from below of the guaranteed values of the policy $\bar{\pi}$. The guaranteed value $J(\bar{\pi})$ for policy $\bar{\pi}$ is rather difficult to compute, but it is possible to compute its reasonable **estimate from below** by the rule $J(\bar{\pi}) \geq J(\bar{\pi}, \bar{w}(\cdot))$, where $\bar{w}(\cdot)$ is some (concrete) admissible disturbance. For example, data from the second row of Table 3 are values of the cost functional $J(\bar{\pi}, \bar{w}(\cdot))$ (see (66)) for the disturbance $\bar{w}(t) = 1/4,\ t \in [0, t_*]$. Hence, there may exist an admissible disturbance $w^*(\cdot)$ such that $J(\bar{\pi}, w^*(\cdot)) > J(\bar{\pi}, \bar{w}(\cdot))$.

Let us remind that the guaranteed value I^0 for policy $\tilde{\pi}^0$ is exact, i.e. there exists an admissible disturbance $w_{\tilde{\pi}^0}(\cdot)$ such that the value of cost functional (4) will be congruent with the estimation. Both policies $\tilde{\pi}^0$ and $\bar{\pi}$ guarantee that at the final moment t_* the system will be in δ_*-neighborhood of a given state z_*.

We can see that the approximative policy $\tilde{\pi}^0$ gives us the guaranteed value of the cost functional (11) better than the estimate $J(\bar{\pi}, \bar{w}(\cdot))$ of the guaranteed value of the cost functional for policy $\bar{\pi}$. Tables 3 and 4 show that the guaranteed value of the cost functional decreases if we add correction points: $I^0(T_k^i) < I^0(T_s^i)$ if $T_k^i \subset T_s^i$. Here T_k^i and T_s^i are some sets of correction points.

For the described policies $\tilde{\pi}^*$ and $\bar{\pi}$, it was interesting to compare the cost function values $J(\tilde{\pi}^*, w(\cdot))$, $J(\bar{\pi}, w(\cdot))$ generated by the same admissible disturbances $w(\cdot)$. Some results of the comparison are presented in Table 5. Table 5 contains the values of the cost functionals for the policies $\tilde{\pi}^*$, $\bar{\pi}$ applied to the dynamic systems (1), (67), A) for the set of correction points T_9^1 for various admissible disturbances.

Let us remind that to construct the policy $\tilde{\pi}^*$, we have to find a solution $\lambda_1^0, ..., \lambda_m^0$ only to the convex problem (60). Using this solution, we determine control laws $\tilde{u}_i^*(t|z_i, v_{i-1})$, $t \in [t_i, t_{i+1} - h]$, by the rules (57), (58) for $i = 0, ..., m-1$ and by the rules (25) for $i = m$.

Table 5. Functions $w(\cdot)$ and the corresponding values of $J(\tilde{\pi}^*, w(\cdot))$ and $J(\bar{\pi}, w(\cdot))$

disturbances $w(\cdot)$	$J(\tilde{\pi}^*, w(\cdot))$	$J(\bar{\pi}, w(\cdot))$
$w_{\tilde{\pi}^*}$	274.211	505.996
$-w_{\tilde{\pi}^*}$	271.340	488.845
α	274.046	504.512
$-\alpha$	271.042	487.381
$\alpha/2$	70.107	129.01
$-\alpha/2$	68.604	120.444
$\alpha cos(t)$	188.164	282.858
$-\alpha cos(t)$	185.637	285.178
$\alpha sin(t)$	86.091	167.238
$-\alpha sin(t)$	84.58	154.313
$\alpha cos(\sqrt{t})$	7.006	7.464
$-\alpha cos(\sqrt{t})$	4.757	2.827
$\alpha sin(\sqrt{t})$	269.03	492.649
$-\alpha sin(\sqrt{t})$	267.113	476.975
0	1.626	0.987
α^2	18.934	34.063
$-\alpha^2$	18.183	29.781

The here presented results of numerical experiments prove that the suggested approximative policy $\tilde{\pi}^*$ is rather good. As shown in Table 5, in the suggested set of admissible disturbances, there are only two disturbances $w(t) = -\alpha cos(\sqrt{t})$ and $w(t) = 0, t \in [0, t_*]$, for which the policy $\tilde{\pi}^*$ gives worse cost functional value than the classical feedback policy $\bar{\pi}$. It is evident that for $w(t) = 0, t \in [0, t_*]$, we get $J(\tilde{\pi}^*, w(\cdot)) > J(\bar{\pi}, w(\cdot))$, because, by construction, the classical feedback policy $\bar{\pi}$ is the best one for the disturbance $w(t) = 0, t \in [0, t_*]$. Note that **there is no** admissible disturbance $w(\cdot)$ with $J(\tilde{\pi}^*, w(\cdot)) > 274.211$ but **there exists** an admissible disturbance $\bar{w}(\cdot)$ with $J(\bar{\pi}, \bar{w}(\cdot)) = 505.996$!

Fig. 1 shows the dependence of the guaranteed values $I^0(h)$ of the cost functional with a delay h for the policy $\tilde{\pi}^0$ applied to the dynamic systems (1) with data (67) and

$$b = (1,1,1,1)^T,\ g = (1,1,1,1)^T,\ \delta_* = 2.456, \tag{69}$$

$$z_0 = (3,3,3,3)^T,\ z_* = -(1,1,1,1)^T,\ \alpha = 1/8, \tag{70}$$

$$T^* = \{t_i = 5i,\ i = 0, ..., 8\},\ t_* = t_8. \tag{71}$$

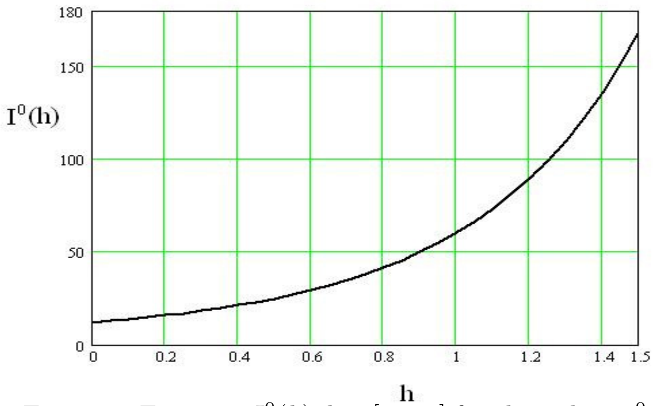

Figure 1. Function $I^0(h), h \in [0, 1.5]$ for the policy $\tilde{\pi}^0$

Fig. 1 shows that the value of delay exerts obvious influence on guaranteed value of the cost functional. Increasing delay value implies increasing of guaranteed value of the cost functional.

For a fixed value $h > 0$ of the system delay and a fixed number m of the correction points, we investigate the dependence of the guaranteed value of the cost function I^0 on a concrete selection of the correction points. Our numerical experiments show that this dependence is essential. For example, let us consider the dynamic systems (1) with the same data (67), (69), (70), and $h = 0.5$, but with different sets of correction points, namely, with the T^* defined in (71) and the following one

$$T^{**} = \{t_0 = 0, t_1 = 5.5, t_2 = 10.5, t_3 = 15.5, t_4 = 20.5, t_5 = 25.5,$$
$$t_6 = 30.5, t_7 = 35, t_8 = t_* = 40\}. \tag{72}$$

As a result we get the guaranteed value of the cost function in the corresponding problems $I^0(T^*) = 24.964$ and $I^0(T^{**}) = 30.189$.

All our numerical experiments were implemented in MATLAB programs.

7. Summary and future work

In this paper, we consider a linear control system with delay, subject to unknown but bounded uncertainties. For this system, we solve the problem of constructing the worst-case feedback control policy, which guarantees that for all admissible uncertainties

- the terminal system state lies in a prescribed neighborhood of a given state z_* at a given final moment,
- the value of the cost function does not exceed a given estimate.

We proposed constructive rules for determining the guaranteed control policy

$$\tilde{\pi}^0 = (\tilde{u}_i^0(\cdot|z_i, v_{i-1}), i = 0, ..., m).$$

The policy consists of control laws

$$\tilde{u}_i^0(\cdot|z_i, v_{i-1}) = (\tilde{u}_i^0(t|z_i, v_{i-1}), t \in T_i),\ i = 0, ..., m.$$

For each $i = 0, ..., m$, the concrete value of law $\tilde{u}_i^0(\cdot|z_i, v_{i-1})$ depends on the current state $z(t_i) = z_i$ of the system at the moments $t = t_i$ and a known control v_{i-1} =$(u(t), t \in \Delta T_i)$ that was constructed at the previous time moment t_{i-1} and that will be applied to the real system at the interval $t \in [t_i, t_i + h]$. Hence, concrete value of the law depends on realized disturbance.

For the policy $\tilde{\pi}^0$, the guaranteed value of the cost function I^0 can be easily found by solving a convex mathematical programming problem with m decision variables.

The proposed control policy $\tilde{\pi}^0$ was tested in numerical experiment. The results of the experiment show that

1) the policy can be easily constructed using standard methods for convex mathematical programming,

2) the policy $\tilde{\pi}^0$ provides the better guaranteed value of the cost functional (11) than a control policy obtained on the basis of a classical feedback that uses the same available current information but does not take into account the corrections in future moments,

3) for the policy $\tilde{\pi}^0$, the guaranteed value of the cost functional decreases if we add new correction points.

Numerical experiments showed that, for a fixed value of delay $h > 0$ and a fixed number m of correction points, the guaranteed value of the cost function essentially depends on a concrete selection of the correction points.

Further research and applications.

1) Due to the fact that the proposed control policy can be easily constructed on-line, it can be effectively used in Model Predictive Control (MPC) if one considers problem (1) as a predictive problem in MPC scheme with moving horizon style.

2) Since for fixed $h > 0$ and m, the guaranteed value of the cost function essentially depends on a concrete selection of the correction points, it will be interesting to investigate the problem of the best selection:

$$\begin{aligned} &\min_{t_1,\dots,\,t_m} \quad I^0(t_1,\dots,t_m) \\ &\text{s.t.} \quad t_{i+1} - t_i \geq h,\ i = 0,\dots,m,\ t_0 = 0, t_{m+1} = t_*. \end{aligned}$$

8. Appendix

Proof of Lemma 2. Let us denote $\bar{z} = z - (F_*F)^{-1}a + F^{-1}G\psi$. Then, problem (38) can be written in the form

$$\begin{aligned} I^0 := \min_{\varphi,\ \psi} \max_{\bar{z}} &(\varphi^T G_* \varphi + \psi^T G \psi + \bar{z}^T \bar{D} \bar{z}) \\ &\text{s.t. } \|\bar{z} - G_*\varphi - F^{-1}G\psi - d\|^2_{Q^{-1}} \leq \sigma, \end{aligned} \tag{73}$$

where $\bar{D}$ and d are defined in (40). Consider the quadratic programming (QP) problem

$$\min_{\varphi,\psi} (\varphi^T G_* \varphi + \psi^T G \psi) \quad \text{s.t. } G_*\varphi + F^{-1}G\psi = c, \tag{74}$$

where $c \in R^n$ is a fixed vector. It follows from optimality conditions for QP problems that if φ and ψ are optimal in problem (74), then there exists a vector $y \in R^n$ such that the following hold true

$$G_*\varphi - G_*y = 0,\ G\psi - GF^{-T}y = 0.$$

Taking these relationships and QP problem (74) into account we conclude that in problem (73) the optimal φ and ψ are related as follows

$$\varphi = y,\ \psi = F^{-T}y \text{ with some } y \in R^n. \tag{75}$$

Substituting (75) in problem (73) we can rewrite the problem (73) in the equivalent form

$$I^0 := \min_{y} \max_{\bar{z}} (y^T \hat{G} y + \bar{z}^T \bar{D} \bar{z}) \quad \text{s.t. } \|\bar{z} - \hat{G}y - d\|^2_{Q^{-1}} \leq \sigma, \tag{76}$$

with $\hat{G} = G_* + F^{-1}GF^{-T}$.

Applying Theorem 1 from Kostina and Kostyukova (2006) to the problem (76) we conclude that the problem (76) is equivalent to the following one

$$\bar{I}^* := \min_{\lambda \geq \mu} \Big(d^T \left(\bar{D}^{-1} + \hat{G} - \frac{Q}{\lambda} \right)^{-1} d + \lambda\sigma \Big) \tag{77}$$

in the sense that $\bar{I}^* = I^0$ and if λ^0 is optimal in (77) then the vector

$$y^0 = -\Big(\bar{D}^{-1} + \hat{G} - \frac{Q}{\lambda^0}\Big)^{-1} d \tag{78}$$

solves the problem (76).

It is easy to see that problems (77) and (39) are the same and that relationships (41) follow immediately from (40), (75), and (78). Lemma 2 is proved. ■

LEMMA 4 *Let $S, A \in R^{n\times n}$ be positive definite matrices and let $K \in R^{n\times n}$ be nonsingular matrix. Then, the matrix $D(\lambda) := S + A - \frac{KK^T}{\lambda}$ is positive definite for all $\lambda \geq \lambda_* := \lambda_{\max}(K^T S^{-1} K)$.*

Proof. Let us prove that the matrix $\bar{D}(\lambda) := S - \frac{KK^T}{\lambda}$ is semi-positive definite for all $\lambda \geq \lambda_*$. It is evident that the matrix $\bar{D}(\lambda)$ is semi-positive definite if and only if the matrix $\tilde{D}(\lambda) := K^{-1}SK^{-T} - I\frac{1}{\lambda}$ is semi-positive definite. In its turn, the matrix $\tilde{D}(\lambda)$ is semi-positive definite if and only if $\lambda_{\min}(K^{-1}SK^{-T}) \geq \frac{1}{\lambda}$ or, equivalently, $\lambda \geq \lambda_{\max}(K^T S^{-1} K)$. Hence, we prove that, for $\lambda \geq \lambda_*$, the matrix $\bar{D}(\lambda)$ is semi-positive definite.

Taking into account the fact that $D(\lambda) = A + \bar{D}(\lambda)$, where the matrix A is positive definite, we conclude that the matrix $D(\lambda)$ is positive definite for all $\lambda \geq \lambda_*$. ■

LEMMA 5 *Consider the matrix $S(\lambda) = (A - \sum_{i=1}^k B_i/\lambda_i)$, $\lambda = (\lambda_1, ..., \lambda_k) \in \Lambda$ where the matrices A, B_i, $i = 1, ..., k$, are positive definite, and the set $\Lambda \subset R^k_+$ is convex. Assume that $S(\lambda)$ is positive definite for all $\lambda \in \Lambda$. Then, the function $\lambda_{\max}(C^T S^{-1}(\lambda) C)$ is convex at Λ, where $C \in R^{n\times r}$, rank $C = r$.*

Proof For any positive definite matrices $\mathcal{A}, \mathcal{B} \in R^{n\times n}$ and a matrix $\mathcal{C} \in R^{n\times r}$, rank$\mathcal{C} = r$, the following relations hold true (see Magnus, Neudecker, 1988)

$$\lambda_{\max}(\mathcal{A}) = 1/\lambda_{\min}(\mathcal{A}^{-1}), \quad \lambda_{\min}(\mathcal{A} + \mathcal{B}) \geq \lambda_{\min}(\mathcal{A}), \tag{79}$$

$$[\mathcal{C}^T(\mathcal{A} + \mathcal{B})^{-1}\mathcal{C}]^{-1} \geq [\mathcal{C}^T\mathcal{A}^{-1}\mathcal{C}]^{-1} + [\mathcal{C}^T\mathcal{B}^{-1}\mathcal{C}]^{-1}. \tag{80}$$

Since $\lambda_{\max}(C^T S^{-1}(\lambda) C) = 1/\lambda_{\min}([C^T S^{-1}(\lambda) C]^{-1})$ we prove the convexity of the function $\lambda_{\max}(C^T S^{-1}(\lambda) C)$ at Λ by showing concavity of the function $q(\lambda) := \lambda_{\min}([C^T S^{-1}(\lambda) C]^{-1})$. Consider the matrix

$$S(\alpha x + (1-\alpha)y) = A - \sum_{i=1}^k \frac{B_i}{\alpha x_i + (1-\alpha) y_i}, \quad x, y \in \Lambda, \ \alpha \in [0,1]. \tag{81}$$

Using the inequality $\dfrac{1}{\alpha a + (1-\alpha) b} \leq \dfrac{\alpha}{a} + \dfrac{1-\alpha}{b}$, for all $a > 0, b > 0, \alpha \in [0,1]$, we may present the matrix (81) in the form

$$
\begin{aligned}
&S(\alpha x + (1-\alpha)y) \\
&= \alpha\Big(A - \sum_{i=1}^{k} B_i/x_i\Big) + (1-\alpha)\Big(A - \sum_{i=1}^{k} B_i/y_i\Big) + \sum_{i=1}^{k} B_i\beta_i \\
&= \alpha S(x) + (1-\alpha)S(y) + \bar{B},
\end{aligned}
\tag{82}
$$

where $\beta_i \geq 0,\ i = 1,..,k$, are some numbers and the matrix $\bar{B} = \sum_{i=1}^{k} B_i\beta_i$ is positive semidefinite. Concavity of the function $q(\lambda), \lambda \in \Lambda$, follows from inequalities (79)-(80) and presentation (82)

$$
\begin{aligned}
&q(\alpha x + (1-\alpha)y) = \lambda_{\min}([C^T S^{-1}(\alpha x + (1-\alpha)y)C]^{-1}) \\
&= \lambda_{\min}([C^T(\alpha S(x) + (1-\alpha)S(y) + \bar{B})^{-1}C]^{-1}) \\
&\geq \alpha\lambda_{\min}[C^T S^{-1}(x)C]^{-1} + (1-\alpha)\lambda_{\min}[C^T S^{-1}(y)C]^{-1} \\
&= \alpha q(x) + (1-\alpha)q(y).
\end{aligned}
\tag{83}
$$

■

LEMMA 6 *The set of feasible solutions to problem (56) is convex.*

Proof. Proof is conducted by induction. Indeed, for $s = m-1$ the set $\Lambda_{m-1} := \{\lambda_m \in R : \mu_m \leq \lambda_m\}$ is convex, and by Lemma 4 the matrix $D_m(\lambda_m)$ is positive definite for all $\lambda_m \in \Lambda_{m-1}$.

Assume that for some index $s,\ i < s \leq m-1$, the set

$$
\begin{aligned}
\Lambda_s := \{&(\lambda_{s+1}, ..., \lambda_m) \in R^{m-s} : \\
&\mu_m \leq \lambda_m,\ \mu_j(\lambda_{j+1}, ..., \lambda_m) \leq \lambda_j,\ j = m-1, ..., s+1\},
\end{aligned}
$$

is convex and the matrix $D_{s+1}(\lambda_{s+1}, ..., \lambda_m)$ is positive definite for all $(\lambda_{s+1}, ..., \lambda_m) \in \Lambda_s$. Consider now index $s-1$. Applying Lemma 5 for the index $k = m-s$ and the matrices $S^{-1}(\lambda) = D_{s+1}(\lambda_{s+1}, ..., \lambda_m)$, $C = F_{s+1}FM_{s-1}^{-1}$ and taking into account the induction assumption we may conclude that the function

$$\mu_s(\lambda_{s+1}, ..., \lambda_m) \text{ for } (\lambda_{s+1}, ..., \lambda_m) \in \Lambda_s$$

is convex. This yields that the set Λ_{s-1} is also convex. Using Lemma 4 it is not difficult to show that the matrix $D_s(\lambda_s, ..., \lambda_m)$ is positive definite for all $(\lambda_s, ..., \lambda_m) \in \Lambda_{s-1}$. Thus, we proved that the sets $\Lambda_s,\ s = m-1, m-2, ..., i$, are convex. To finish the proof we need to notice that the set of feasible solutions to the problem (56) coincides with Λ_i. ■

LEMMA 7 *The cost function of the problem (56) is convex.*

Proof. Proof follows from Lemma 6 and Lemma 5 with $S^{-1}(\lambda) = D_{i+1}(\lambda_{i+1}, ..., \lambda_m)$ and $C = d_{i+1}(z_i, v_{i-1}) \in R^{n\times 1}$. ■

Acknowledgment

We are grateful to the anonymous referee for precise and substantial remarks which led to an essentially improved version of the paper.

References

Basin, M. and Rodrigez-Gonzalez, J. (2005) A closed-form optimal control for linear system with equal state and input delays. *Computers & Structures* **41**(5), 915–920.

Bellman, R. (1961) *Adaptive Control Processes – A Guided Tour.* Princeton University Press.

Bemporad, A., Borrelli, F. and Morari, M. (2003) Min-Max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control* **48**(9), 1600–1606.

Boukas, E. and Al-Muthairi, N.F. (2006) Delay-dependent stabilization of singular linear systems with delay. *International Journal of Innovative Computing Information and Control* **2**(2), 283-291.

Kerrigan, E. and Maciejowski, J.M. (2004) Feedback min-max model predictive control using a single linear program: Robust stability and explicit solution. *Int. J. of Robust and Nonlinear Control* **14**, 395–413.

Kim, J.H. (2000) Guaranteed cost control of parameter uncertain systems with time delay. *Transaction on Control, Automation, and Systems Engineering* **2** (1), 19–23.

Kothare, M.V., Balakrishnan, V. and Morari, M. (1996) Robust constrained model predictive control using linear matrix inequalities. *Automatica* **32**(10), 1361–1379.

Kostina, E.A. and Kostyukova, O.I. (2006) Robust optimal feedback for terminal linear-quadratic control problems under disturbances. *Mathematical Programming, Ser. B* **107**, 131–153.

Langson, W., Chryssochoos, I., Rakovic, S.V. and Mayne, D.Q. (2004) Robust model predictive control using tubes. *Automatica* **40**, 125–133.

Lee, J.H. and Yu, Z. (1997) Worst-case formulation of model predictive control for systems with bounded parameters. *Automatica* **33**(5), 763–781.

Magni, L., De Nicolao, G., Scattolini, R. and Allgöwer, F. (2002) Robust receding horizon control for non-linear discrete-time system. In: *Proceedings of the 15th IFAC World Congress*, Spain.

Magnus, J.R. and Neudecker, H. (1988) *Matrix Differential Calculus with Applications in Statistics and Econometrics.* Wiley and Sons.

Mahmoud, M.S. (2000) *Robust Control and Filtering for Time-Delay Systems.* Marcel Dekker.

Matveev, A.S. and Yakubovich, V.A. (1998) Nonconvex global optimization in optimal control theory. *Itogi nauki i tekhniki, Seriya Sovremennaya matematika i ee prilozheniya* **60**, 128–175.

Pontryagin, L.S., Boltyanskij, V.G., Gamkrelidze, R.V. and Mishchenko, E.F. (1986) *Selected Works, Vol. 4, The Mathematical Theory of Optimal Processes.* Gamkrelidze, R.V. (ed.) Classics of Soviet Mathematics, Gordon and Breach Science Publishers, New York. (Translation from Russian edition, Nauka, Moscow, 1961).

Richard, J.-P. (2003) Time-delay systems: an overview of some recent advances and open problems. *Automatica* **39**, 1667–1694.

Savkin, A., Skafidas, E. and Evans, R. (1999) Robust output feedback stabilizability via controller switching. *Automatica* **35**, 69–74.

Scokaert, P.M and Mayne, D.Q. (1998) Min-max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic Control* **43** (8), 1136-1142.

Shi, P. Boukas, E., Shi, Y. and Agarwal, R.K. (2003) Optimal guaranteed cost control of uncertain discrete time-delay systems. *Journal of Computational and Applied Mathematics* **157**, 435-451.

Vanderberghe, L., Boyd, S. and Nouralishahi, M. (2002) Robust linear programming and optimal control. In: *Proceedings of the 15th IFAC World Congress on Automatic Control*, Spain.

Yu, L. and Gao, F. (2001) Optimal guaranteed cost control of discrete-time uncertain systems with both state and input delays. *Journal of the Franklin Institute* **338**, 101–110.