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Shifts of the term structure of interest rates against which a given portfolio is preimmunized*

by

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Abstract: In this paper we formulate an immunization problem, which is rarely stated. Instead of reconstructing an existing bond portfolio B with the aim of securing a desired amount of, say L dollars, q years from now, against uncertain future interest rates shifts (under various, sometimes strong assumptions), we identify the shifts of the current term structure of interest rates against which portfolio B is already preimmunized. We state this problem in two different mathematical settings, and solve it with the help of Proposition 2 from Barber (1999), or, equivalently, Theorem 1 from Rzadkowski and Zaremba (2000). In the first part of this paper shifts are supposed to be polynomials of degree less than a certain number n, while in the second part, where we employ a Hilbert space approach, the shifts are allowed to be continuous functions.

Keywords: immunization, term structure of interest rates, polynomial shifts, Hilbert space approach.

1. Problem formulation

Suppose a decision maker possessing C dollars today must achieve an investment goal of $L > C$ dollars q years from now by means of a purchase of an appropriately selected bond portfolio B. If not successful he/she will incur a severe penalty, while achieving more than L dollars will result in no rewards. Such investors are said to be bond immunizers. Several strategies aimed at the construction of such bond portfolio B have been advocated for immunization purposes (see references).

By the term structure of interest rates (TSIR) one understands a schedule of spot interest rates. The term structure as a function, say $s(t)$, can be flat, rising, declining, or humped. Analysts try to estimate it from the yields for

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coupon-bearing bonds. We will be concerned with discrete time models, when either coupons or par values, to be denoted thereafter by c_i, are payable at some instances $t_i \in [0, T]$. If $PV(t)$ stands for the present value of a zero-coupon bond with the par value of 1 dollar maturing at time t (after t years), then the formula

$$PV(t) = e^{-s(t)t} \tag{1}$$

holds, provided interest rates are continuously compounded. The mapping $t \rightarrow PV(t)$ is said to be a discount function, while $e^{-s(t)t}$ are called discount factors. Let

$$s^*(t) = s(t) + \lambda a(t) \tag{2}$$

be a new yield curve, which is the result of changes in bond prices caused by various market forces. The random parameter λ, whose probability distribution does not play any role in our approach, represents the unknown today magnitude of the shifts to occur in TSIR, while $a(t)$ stands for the postulated shifts. In this paper $a(t)$ is not an apriori specified function as is usually the case, but is allowed to be any shift from a specified class of functions, either polynomials (Section 2) or continuous functions (Section 3).

Our goal is to identify those shifts $a(t)$ in the current TSIR, against which the value of a given portfolio B, which we either already possess or are going to purchase today, is immunized q years from now for all λ.

2. Model 1: shift can be any polynomial

Suppose we possess a bond portfolio B consisting of bonds generating payments c_i at instances t_i, $i = 1, 2, \ldots, m$. A liability of L dollars has to be discharged at a future date q by means of B irrespective of shifts in the TSIR, which may take place in the meantime, as long as the new TSIR is of the form (2). The immunization means that if $FVB(t)$ stands for the future value of B at time t, then $FVB(q) \geq L$, that is, the value of B at time q will be no less than the liability to be paid off at time q.

Theorem 2.1 *Assume that shifts in the TSIR are of the form*

$$a(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ... + a_{n-1} t^{n-1} = \sum_{j=1}^{n} a_{j-1} t^{j-1} .$$

The family of those polynomials $a(t)$ of the form given above, which ensure the immunization at time q is an $(n-1)$ dimensional linear subspace (denoted thereafter by IMMU) of the space of all polynomials of degree $\leq (n-1)$, which itself has dimension n.

Before proving this theorem, let us remark that under the current TSIR, which we are denoting by $s(t)$, the inequality $FVB(q) \geq L$ can be rewritten as

$$FVB(q) = e^{s(q)q} PV(B) = \sum_{i=1}^{m} c_i e^{s(q)q - s(t_i)t_i} \geq L, \tag{3}$$

where $PV(B)$, standing for the present value of portfolio B, is given by the formula

$$PV(B) = \sum_{i=1}^{m} c_i e^{-s(t_i)t_i}. \tag{4}$$

Classical results (proved under simplifying assumptions) assert that immunization takes place if the so called portfolio "duration" T_P is equal to q, where

$$T_P = \sum_{i=1}^{m} w_{t_i} t_i. \tag{5}$$

Here the weights w_{t_i} of payoffs c_i are given either by

$$w_{t_i} = \frac{c_i}{[1+s(t_i)]^{t_i}} / PV(B) \quad \text{or by} \quad w_{t_i} = c_i e^{-s(t_i)t_i} / PV(B) \tag{6}$$

depending on the way the interest rate is compounded.

Definition 2.1 *A set S of vectors/elements is said to be a linear space if the sum of arbitrary two elements $a \in S$ and $b \in S$ belongs to S ($a + b \in S$), and for any real number r the product of r and a belongs to S as well ($ra \in S$).*

Proposition 2 from Barber (1999), as well as Theorem 1 from Rzadkowski and Zaremba (2000), says that immunization of portfolio B at time q will be secured provided the following sufficient condition holds

$$a(q)q \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} = \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} a(t_i) t_i. \tag{7}$$

Fact 2.1 *The family of all piecewise continuous shocks $a(t)$ satisfying* (7) *is a linear space. Similarly, the family of all continuous functions (or polynomials of degree less than an arbitrary natural number k), which satisfy* (7) *is a linear space.*

Proof of Theorem 2.1 Let us start with the observation that condition (7) is a generalization of the mentioned above classical immunization result (corresponding to $a(t) \equiv 1$) which claims that immunization holds if $T_P = q$, where T_P is given by (5) and (6). Our aim is to identify and characterize all polynomials of degree $\leq (n-1)$ satisfying (7). We know from Fact 2.1 that these polynomials constitute a linear subspace, say IMMU. Once we identify all functions belonging to IMMU, we will know which polynomials (shifts) the portfolio B is already immunized against. Since all shifts in the TSIR are of the form

$$a(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ... + a_{n-1} t^{n-1} = \sum_{j=1}^{n} a_{j-1} t^{j-1}, \tag{8}$$

we substitute (8) into (7) to arrive at

$$LS = (\sum_{j=1}^{n} a_{j-1} q^j) \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} = \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} (\sum_{j=1}^{n} a_{j-1}(t_i)^j) = RS, \quad (9)$$

where $LS(RS)$ stand for the value of the left (respectively right) hand side of the above equality. RS can be rearranged, by first fixing index j, and then multiplying a_{j-1} by all terms dependent on index i. The result of this will be the equality

$$RS = \sum_{j=1}^{n} a_{j-1} (\sum_{i=1}^{m} c_i e^{-s(t_i)t_i} (t_i)^j). \quad (10)$$

Next, by subtracting RS from LS we get a single linear equation

$$0 = LS - RS = \sum_{j=1}^{n} a_{j-1} \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} [q^j - (t_i)^j] = \sum_{j=1}^{n} a_{j-1} A_{j-1}, \quad (11)$$

where

$$A_{j-1} = \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} [q^j - (t_i)^j],$$

with n unknown variables $a_0, a_1, ..., a_{n-1}$, which may naturally be viewed as elements of R_n, the latter being n-dimensional linear space. Let us note that A_{j-1} depend solely on the cash flow generated by portfolio B and the parameters t_i determined by the market. The well known in matrix algebra Kronecker-Capelli theorem applied to (11) asserts that the set of solutions $a_0, a_1, ..., a_{n-1}$ of (11) constitutes an $(n-1)$-dimensional linear space. In this way we have proved that IMMU consists of all polynomials of the form (8), whose coefficients $a_0, a_1, ..., a_{n-1}$ belong to this linear space, and consequently IMMU is an $(n-1)$-dimensional subspace of the space of all polynomials of degree $\leq (n-1)$.

Definition 2.2 *A set of vectors $l_1, l_2, ...l_m$ from a linear space S is said to be linearly independent if $\alpha_1 l_1 + \alpha_2 l_2 + ... + \alpha_m l_m \neq 0$ whenever real numbers $\alpha_1, \alpha_2, ...\alpha_m$ are not all equal to zero.*

Definition 2.3 *A set of linearly independent vectors $l_1, l_2, ...l_m$ from a linear space S is said to be a base for S if each element (vector) of S is a linear combination of $l_1, l_2, ...l_m$, and this property does not hold any longer after removal of any of the vectors l_i.*

Fact 2.2 *Each base for a k-dimensional linear subspace S must be a set of k linearly independent vectors, and conversely, each set of k linearly independent vectors belonging to S is a base for S.*

FACT 2.3 *If $A_{j-1} \neq 0$ for $j = 1, 2, \ldots, n$, then the following polynomials*

$$a_1(t) = \frac{-A_1}{A_0} + t, a_2(t) = \frac{-A_2}{A_0} + t^2, \ldots,\ a_{n-1}(t) = \frac{-A_{n-1}}{A_0} + t^{n-1}, \quad (12)$$

constitute a base for the subspace IMMU.

Proof. Based on Theorem 2.1 and Fact 2.2, it is enough to demonstrate that each of these $(n-1)$ polynomials solves (11) and that these polynomials are linearly independent, the latter being a trivial observation. Let us assume that $a_2 = a_3 = ... = a_{n-1} = 0$ and next solve (11) for a_0, a_1. Let us first notice that $a_1 \neq 0$ because otherwise a_0 would have to be equal to zero due to the inequality $A_0 \neq 0$, and the equation $\sum_{j=1}^{n} a_{j-1}A_{j-1} = 0$. Seeking for a non-zero solution of (11), we may assume without loss of generality that $a_1 = 1$, and hence $a_0 = -A_1/A_0$, which implies that polynomial $a_1(t)$, given above, solves (11). To show that the specified above polynomial $a_2(t)$ is also a solution to (11), we argue similarly, supposing that $a_1 = 0$ as well as $a_3 = a_4 = a_5 = ... = a_{n-1} = 0$, which leads to a linear equation for a_0, a_2, whose solution will appear to be the polynomial $a_2(t)$. In the same manner we demonstrate that all remaining polynomials are solutions to (11). ∎

3. Model 2: shift can be any continuous function defined on $[0, T]$

This time, our aim is to identify all continuous shifts/shocks $a(t)$ to the TSIR, which our portfolio B is already immunized against with the new term structure of the form (2). We start with a definition of Hilbert space, naming it H. As such, H must be a linear space of vectors/elements, that is, a set of elements that can be summed up and multiplied by a scalar without leaving the set. Secondly, H must be equipped with a norm and a scalar product of two arbitrary vectors from H. Let us define H as the set of all continuous functions defined on the interval $[0, T]$, representing the life span for bonds available on a given debt market. Given two elements of H, that is, two continuous functions $f(t)$ and $g(t)$, defined on $[0, T]$, let us define their scalar product as

$$\langle f, g \rangle = \sum_{i=1}^{m} c_i e^{-s(t_i)t_i} t_i f(t_i) g(t_i). \quad (13)$$

The norm of an arbitrary element $f \in H$ must then be defined as $\|f\| = \sqrt{\langle f, f \rangle}$, the latter implying that $\|f\| = 0$ if and only if $f(t_i) = 0$ for each t_i, $i = 1, 2, \ldots, m$. Two functions, $f(t)$ and $g(t)$ are identical as elements of H, when $\|f - g\| = 0$, that is, $f(t_i) = g(t_i)$ for all instances t_i when portfolio B generates payments. Our nearest goal is to determine a base in H consisting of orthonormal polynomials $P_k(t)$ of degree k, where $k = 0, 1, 2, ..., m-1$. It means that all of them will have length 1 and be mutually perpendicular, that is, $\|P_k(t)\| = 1$ and $\langle P_k(t), P_l(t)\rangle = \delta_{kl}$, with $\delta_{kl} = 0$ for $k \neq l$ and $\delta_{kl} = 1$ for $k = l$. If we do this, each element of H, that is, each continuous function $a(t)$ will be identifiable with a certain linear combination $a^*(t)$ of base polynomials $P_k(t)$. One will then have $\|a(t) - a^*(t)\| = 0$ and

$$a(t) \approx a^*(t) = a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + ... + a_{m-1}P_{m-1}(t). \tag{14}$$

The identification above means that those two functions coincide at all instances t_i. Let us underline that as of that moment we do not know these polynomials, but later on we will say how to determine them. Similarly as in Section 2, portfolio B will be immunized under the new TSIR of the form $s^*(t) = s(t) + \lambda a(t)$ if condition (7) is satisfied. To make sure it is, we substitute $a^*(t)$ for $a(t)$ into (7), to obtain the relationship

$$\begin{aligned}&[a_0P_0(q) + a_1P_1(q) + ... + a_{m-1}P_{m-1}(q)]q\sum_{i=1}^{m} c_ie^{-s(t_i)t_i}\\ &= \sum_{i=1}^{m} c_ie^{-s(t_i)t_i}a^*(t_i)t_i.\end{aligned} \tag{15}$$

The right hand side of (3) can be substantially simplified. As a matter of fact, since polynomials $P_i(t)$, $0 \leq i \leq m-1$, are mutually orthogonal, the first of them, $P_0(t)$, which is a polynomial of degree zero, has to be orthogonal to $P_1(t), P_2(t), ..., P_{m-1}(t)$, which implies $P_1(t), P_2(t), ..., P_{m-1}(t)$ are also orthogonal to the function identically equal to 1, that is,

$$\langle a_jP_j(t),\, 1\rangle = \sum_{i=1}^{m} c_ie^{-s(t_i)t_i}[a_jP_j(t_i)1]t_i = 0, \qquad j = 1, 2, ..., m-1, \tag{16}$$

the latter significantly simplifying Eq. (3) because the right hand side of (3) will then reduce to the number

$$\sum_{i=1}^{m} c_ie^{-s(t_i)t_i}a_0P_0t_i,$$

leading consequently to the equation

$$\begin{aligned}&[a_0P_0 + a_1P_1(q) + ... + a_{m-1}P_{m-1}(q)]q\sum_{i=1}^{m} c_ie^{-s(t_i)t_i}\\ &= \sum_{i=1}^{m} c_ie^{-s(t_i)t_i}t_i(a_0P_0).\end{aligned} \tag{17}$$

Using next the so-called Gram-Schmidt orthogonalization procedure (Example 2 shows how this method works) one can determine polynomials $P_i(t)$, $0 \leq i \leq m-1$. After having done this, (3) becomes a linear equation with m unknown coefficients $a_0, a_1, ..., a_{m-1}$, whose solution gives rise to an $(m-1)$ dimensional subspace of coefficients. In this way we have proven the theorem below.

Theorem 3.1 *Suppose a bond portfolio B is given and shifts in the TSIR are continuous functions defined on $[0, T]$. Then the set of these shifts equipped with the scalar product defined by (14) constitutes an m-dimensional Hilbert space, where m is the number of instances when portfolio B generates payments. The subset of those shifts, which portfolio B is already immunized against at time q is an $(m-1)$-dimensional subspace (depending on B) of the form*

$$a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + \ldots + a_{m-1}P_{m-1}(t), \tag{18}$$

where the m linearly independent polynomials $P_i(t)$, $i = 0, 1, ..., m-1$ constitute a base which may be determined by the Gram-Schmidt orthogonalization procedure, while the coefficients $a_0, a_1, ..., a_{m-1}$ can be found as solutions of Eq. (17).

In practical terms this means that bond portfolio B is immunized against a shift $a(t)$ if $\|a(t) - a^*(t)\| = 0$ holds for some $a^*(t) = a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + ... + a_{m-1}P_{m-1}(t)$. We know that $a(t)$ coincides with $a^*(t)$ at all instances t_i when B generates its payments.

4. Examples

Example 4.1 (Shift can be any polynomial of degree ≤ 4)

Let the TSIR be of the form $s(t) = 0.065 - 0.0005t$ for $0 \leq t \leq 5$ with shifts being polynomials

$$a(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4. \tag{19}$$

Let our portfolio B reduce to a single bond which pays 5 coupons $c_i = 10$ at instances $t_i = i$ with $i = 1, 2, 3, 4, 5$, and the par value of $c_5 = 100$ at the maturity ($t_5 = 5$). Let moreover our liability of L dollars (the present value of L is equal to the present value of B) have to be discharged $q = 4.5$ years from now. Based on (11) the coefficients a_{j-1} of each shift $a(t) = \sum_{j=1}^{5} a_{j-1}t^{j-1}$, against which our bond B is "automatically" preimmunized $q = 4.5$ years from now must fulfill the linear equation

$$30.86a_0 + 67.2a_1 - 409a_2 - 6078.1a_3 - 50116.87a_4 = 0, \tag{20}$$

which leads to the base polynomials

$$a_1(t) = -2.18 + t,\; a_2(t) = 13.26 + t^2,\; a_3(t) = 196.96 + t^3,\; a_4(t) = 1624.01 + t^4, \tag{21}$$

being of the form (12) according to Fact 2.3.

Example 4.2 (Shift can be any continuous function defined on $[0, T]$)

Let the TSIR, bond B and liability L be the same as in Example 4.1. Theorem 3.1 asserts that the subset of these shifts against which bond B is already preimmunized at time $q = 4.5$ is a 4-dimensional subspace of continuous functions H of the form

$$a^*(t) = a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + ... + a_{5-1}P_{5-1}(t), \tag{22}$$

where polynomials $P_i(t)$,$i = 0, 1, 2, 3, 4$, may be determined by the Gram-Schmidt orthogonalization procedure, while the coefficients $a_0, a_1, ..., a_{m-1}$ can be found as solutions of Eq. (17). Let us therefore find a base consisting of five polynomials $P_k(t)$ of degree k ($k = 0, 1, 2, 3, 4$), which satisfy

$$\begin{aligned}&\langle P_k(t), P_l(t)\rangle = \delta_{kl},\ 0 \le k, l \le 4,\\ &\delta_{kl} = 0 \ \text{ for } k \ne l,\ \delta_{kl} = 1 \ \text{ for } k = l\ (0 \le k \le 4).\end{aligned} \tag{23}$$

In order to determine polynomial P_0 (of degree zero), we make use of the relationship $\langle P_0, P_0\rangle = 1$ occurring in (23). Having determined P_0, we identify polynomial P_1 of degree 1 with two unknown coefficients, by making use of the relationships $\langle P_1, P_1\rangle= 1$ and $\langle P_0, P_1\rangle = 0$ occurring in (23). Knowing P_0 and P_1, we are in a position to identify polynomial P_2 with three unknown coefficients, by means of the three relationships:

$$\langle P_2, P_2\rangle = 1, \quad \langle P_2, P_1\rangle = 0, \quad \langle P_2, P_0\rangle = 0. \tag{24}$$

Proceeding in this way with the help of a Solver, one arrives at the polynomials

$$\begin{aligned}P_0(t) &= 0.04721\\ P_1(t) &= 0.23899 - 0.05174t\\ P_2(t) &= 0.52437 - 0.36629t + 0.05267t^2\\ P_3(t) &= -1.14161 + 1.46274t - 0.5215t^2 + 0.05496t^3\\ P_4(t) &= 3.39874 - 6.23373t + 3.70313t^2 - 0.8784t^3 + 0.07199t^4.\end{aligned} \tag{25}$$

Now one can rewrite Eq. (17) in the form

$$1.62941a_0 + 2.97331a_1 - 27.7354a_2 - 53.8486a_3 - 91.2228a_4 = 0. \tag{26}$$

Based on Theorem 3.1, the set of shifts against which B is preimmunized $q = 4.5$ years from now consists of all functions of the form

$$a^*(t) = a_0P_0(t) + a_1P_1(t) + a_2P_2(t) + ... + a_{5-1}P_{5-1}(t), \tag{27}$$

where the polynomials $P_i(t)$, $i = 0, 1, 2, 3, 4$, are given by (25), while $a_0, a_1, ..., a_{m-1}$ satisfy Equation (26). In fact, P is immunized against each shift $a = a(t)$ if $\|a - a^*\| = 0$ for some $a^*(t)$ described by (27), what means that $a(t)$ coincides with $a^*(t)$ at all instances $t_i = 1, 2, 3, 4, 5$..

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