Control and Cybernetics

vol. **39** (2010) No. 3

Two variations of the Public Good Index for games with a priori unions*

by

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Abstract: This paper introduces two variations of the Public Good Index (Holler, 1982) for games with a priori unions. The first one stresses the public good property which suggests that all members of a winning coalition derive equal power. The second variation follows earlier work on the integration of a priori unions (Owen, 1977 and 1982) and refers to essential subsets of an a priori union when allocating power shares. Axiomatic characterizations of both indices are discussed. Numerical examples, one of them taken from a political setting, illustrate the new power indices presented in this paper.

Keywords: simple game, coalition structure, Public Good Index.

1. Introduction

The classical model of cooperative games with transferable utility (TU-games, to abbreviate), involves a set of players in such a way that if a coalition of them decides to cooperate, they can guarantee a certain payoff. To share the payoff of the total coalition, different solutions were studied in the literature. One of the most important of these solutions is the Shapley value (1953).

With the passing of time, due to the complexity of many real situations, the traditional model of TU-games was enriched. Aumann and Drèze (1974) and Owen (1977) considered TU-games where there is a system of unions among the players, which is formed previously to the negotiation process. Aumann

*Submitted: October 2008; Accepted: November 2009.

and Drèze's value for TU-games with a priori unions is defined with the condition of no transference of payoff among the unions, that is, each union only obtains the payoff that it can guarantee itself. The so-called Owen value is a two-step extension of the Shapley value that takes a priori unions into consideration. In the first step, i.e., in the induced game played by the a priori unions (quotient game), this measure distributes the total value among the unions in accordance with the Shapley value. In the second step, once again applying the Shapley value, the total reward of a union is allocated among its members taking into account the possibility of joining other unions. The Owen value is a coalitional value of the Shapley value: it coincides with the Shapley value if each a priori union contains one element only. One of its most appealing properties is the property of symmetry in the quotient game: given two unions which play symmetric roles in the quotient game, they are awarded with the same apportionment of the total payoff.

An important family of TU-games with transferable utility is formed by the so-called simple games. Simple games can be used to model decision-making processes. Many interesting applications have been developed in the field of political science, since voting rules in Parliaments or in committees can be modeled as simple games. An important issue in this setting is to measure agents' power. The literature offers a series of alternative measures called power indices: the Shapley-Shubik index (Shapley and Shubik, 1954), which is the restriction of the Shapley value to simple games, the Banzhaf-Coleman index (Banzhaf, 1965; Coleman, 1971), the Deegan-Packel (DP) index (Deegan and Packel, 1978), and the Johnston index (Johnston, 1978). In this paper we will focus on the Public Good Index (PGI). This measure was first applied in Holler (1978), then explicitly proposed in Holler (1982) and axiomatized in Holler and Packel (1983).

The addition of a system of a priori unions plays an important role in the family of simple games: deputies belong to parties, members of a committee belong to corporations, and so on. Some of the previous indices have a counterpart in this context. The Owen value provides a counterpart of the Shapley-Shubik index; the Banzhaf-Owen value (Owen, 1982; Albizuri, 2001; Amer et al., 2002 and Alonso-Meijide et al., 2007) is a counterpart for the Banzhaf-Coleman index; the symmetric coalitional Banzhaf value (Alonso-Meijide and Fiestras-Janeiro, 2002) is a counterpart for both the Shapley-Shubik and the Banzhaf-Coleman index; and the coalitional Deegan-Packel index (Alonso-Meijide et al., 2009) is a counterpart for the Deegan-Packel index.

In this paper, we will introduce two extensions of the PGI for simple games with a priori unions. The first one stresses the public good property, which suggests that all members of a winning coalition derive equal power, irrespective of their possibility to form alternative coalitions. In games with a priori unions it seems "natural" to apply the notion of decisiveness and the concept of minimal winning coalition to the quotient game only. Partners in an a priori union cannot be excluded from enjoying the coalition value, but, as well, partners cannot absent themselves from the costs implied by an a priori union. The second extension follows earlier work on the integration of a priori unions (see Owen, 1977 and 1982). It refers to essential subsets when allocating power shares, taking the outside options of the coalition members into consideration.

Axiomatic characterizations of both indices are discussed. Numerical examples, one of them taken from a political setting, illustrate the new power indices. Obviously, the two versions constitute different solution concepts. The discussion would demonstrate again that "different solution concepts can therefore be thought of as results of choosing not only which properties one likes, but also which examples one wishes to avoid" (Aumann, 1977, p.471).

The paper is organized as follows. In Section 2, we introduce the analytical tools and restate some basic definitions. In Section 3, using the principle of solidarity inside unions, we define and characterize a first extension of the Public Good Index. In Section 4, we define and characterize a second extension, following a similar procedure to that of Owen. Finally, we illustrate and compare these extensions using a real-world example.

2. Preliminaries

2.1. Simple games

A **simple game** is a pair (N, W) where N is the set of players and W is a set of subsets of N satisfying:

- $N \in W$, $\emptyset \notin W$ and
- the monotonicity property, *i.e.*,

$$S \subseteq T \subseteq N \text{ and } S \in W \text{ implies } T \in W.$$

This representation of simple games follows the approach by Felsenthal and Machover (1998) and by Peleg and Sudhölter (2003). Intuitively, N is the set of members of a committee and W is the set of winning coalitions. We denote by $SI(N)$ the set of simple games with player set N.

Take a simple game (N, W). A coalition S is any subset of the set of players N. For each $i \in N$ and $S \subseteq N$, we will use shorthand notation and write $S \cup i$ for the set $S \cup \{i\}$ and $S \backslash i$ for the set $S \backslash \{i\}$. A coalition $S \subseteq N$ is **winning** if $S \in W$ and is **losing** if $S \notin W$. A winning coalition $S \in W$ is a **minimal winning coalition** (MWC) if every proper subset of S is a losing coalition, that is, S is a MWC in (N, W) if $S \in W$ and $T \notin W$ for any $T \subset S$. We denote by M^W the set of MWC of the simple game (N, W). From the monotonicity property, a simple game is clearly defined by the set of players and the set of minimal winning coalitions. Given a player $i \in N$ we denote by M_i^W the set of MWC such that i belongs to, that is, $M_i^W = \{S \in M^W / i \in S\}$. A player $i \in N$ is a **null player** if and only if $i \notin S$ for all $S \in M^W$. Two players $i, j \in N$ are **symmetric** in a simple game (N, W) if $S \cup i \in W$ if and only if $S \cup j \in W$ for all $S \subseteq N \backslash \{i, j\}$. Given a coalition $T \subseteq N$, the **unanimity game** of coalition T, (N, W_T), is the simple game with $M^{W_T} = \{T\}$.

A **power index** is a function f which assigns a non-negative n-dimensional real vector $f(N,W)$ to a simple game (N,W), where the i-th component of this vector, $f_i(N,W)$, is the power of player i in the game (N,W) according to f.

Here we estate three well-known properties that some power indices satisfy.

A power index f satisfies *efficiency* if and only if for every simple game (N,W), $\sum_{i\in N} f_i(N,W)=1$.

A power index f satisfies the *null player* property if and only if for every simple game (N,W) and a null player $i\in N$, $f_i(N,W)=0$.

A power index f satisfies *symmetry* if and only if for every simple game (N,W), and $i,j\in N$ symmetric players in the game, $f_i(N,W)=f_j(N,W)$.

2.2. The Public Good Index

Holler (1982) proposed the Public Good Index (PGI). The primary application of the PGI was to analyze situations, in which voting decisions on the selection of a public good is considered. In the computation of the PGI, the MWC are the only relevant coalitions. It is assumed that coalitions that are not MWC do not matter, and thus should not be taken into consideration, when it comes to measuring power. That is, although only MWC are taken into account for the calculation of the PGI, it is not said that no other coalitions will be formed.

Given a simple game (N,W), the PGI assigns to each player $i\in N$ the real number

$$\delta_i(N,W)=\frac{\left|M_i^W\right|}{\sum_{j\in N}\left|M_j^W\right|}. \tag{1}$$

That is, the PGI of a player i is equal to the total number of MWC containing player i, normalized by the sum of these numbers for all players.

An axiomatic characterization of this index can be found in Holler and Packel (1983). The characterization used in that paper applies the properties of symmetry, efficiency, null player, and PGI- mergeability. To specify the latter property, we introduce the definition of mergeable games.

Mergeable games. Given two simple games $(N,W),(N,V)$, the simple game $(N,W\vee V)$ is defined in such a way that a coalition $S\in W\vee V$ if $S\in W$ or $S\in V$. Two simple games (N,W) and (N,V) are mergeable if for any $S\in M^W$ and for any $T\in M^V$, $S\nsubseteq T$ and $T\nsubseteq S$.

The mergeability condition guarantees that the set of MWC of the game $(N,W\vee V)$ is the union of the MWC sets of the games (N,W) and (N,V) (when (N,W) and (N,V) are two mergeable games). Then, $\left|M^{W\vee V}\right|=\left|M^W\right|+\left|M^V\right|$. Mergeability refers to the possibility that two voting bodies are involved in a collective decision. Note that two-chamber parliamentary systems are quite common. Take for instance the US Senate and House, the German Bundestag and Bundesrat or EU codecision making which involves the Council of Ministers and the European Parliament. In general, it needs the approval of both chambers for acceptance of decisions under such a regime. The mergeability condition presupposes that the consent of one chamber is sufficient to let a motion successfully pass. To some extent this reflects political reality. If one chamber is in favor of motion, then quite often the second supports this decision, although contradicting its own preferences, in order to avoid conflict and to hope for a similar support in a reciprocal situation. It has been said that this is how the conciliation committees in codecision making work.

PGI-mergeability. A power index f satisfies PGI-mergeability if for any pair of mergeable games $(N, W), (N, V)$, it holds that

$$f(N, W \vee V) = \frac{\sum_{j \in N} \left|M_j^W\right|}{\sum_{j \in N} \left|M_j^{W \vee V}\right|} f(N, W) + \frac{\sum_{j \in N} \left|M_j^V\right|}{\sum_{j \in N} \left|M_j^{W \vee V}\right|} f(N, V). \quad (2)$$

That is, power in a merged game is a weighted mean of the power of the component games, with the sum of the number of MWC containing each player providing the weights.

Theorem 1 (Holler and Packel, 1983) *The PGI is the unique power index defined on $SI(N)$ satisfying efficiency, null player, symmetry, and PGI-mergeability.*

Alternatively, Alonso-Meijide et al. (2008) characterized the PGI replacing the property of PGI-mergeability with the property of PGI-minimal monotonicity. It takes into account a relation between two simple games (N, W) and (N, V) given in terms of cardinality of the sets of MWC.

PGI-minimal monotonicity. A power index f satisfies PGI-minimal monotonicity if for any pair of simple games (N, W), (N, V), it holds that for every player $i \in N$ such that $M_i^W \subseteq M_i^V$,

$$f_i(N, V) \sum_{j \in N} \left|M_j^V\right| \geq f_i(N, W) \sum_{j \in N} \left|M_j^W\right|.$$

That is, if the set of MWC containing a player $i \in N$ in game (N, W) is a subset of the set of MWC containing this player in game (N, V), then the power of player i in game (N, V) is not less than power of player i in game (N, W), once this power is normalized by the sum over all players of the total number of MWC containing each player in both games (N, W) and (N, V).

Theorem 2 (Alonso-Meijide et al., 2008) *The PGI is the unique power index f defined on $SI(N)$ satisfying efficiency, null player, symmetry, and PGI-minimal monotonicity.*

2.3. Games with a priori unions

Given a finite set of players N, we will denote by $P(N)$ the set of all partitions of N. An element $P \in P(N)$ is called a coalition structure: it describes the a priori unions on N. A simple game with a priori unions is a triple (N, W, P), where (N, W) is a simple game and $P \in P(N)$. We denote by $SIU(N)$ the set of simple games with a priori unions and player set N.

Given $(N, W, P) \in SIU(N)$, with $P = \{P_1, P_2, \ldots, P_u\}$, the **quotient game** is the simple game $(U, \overline{W})$, where the set of players $U = \{1, \ldots, u\}$ are the unions. A set $R \subseteq U$ is a winning coalition in $(U, \overline{W})$ if $\bigcup_{k \in R} P_k$ is a winning coalition in (N, W). Two unions $P_k, P_s \in P$ are symmetric if k and s are symmetric players in $(U, \overline{W})$.

Take a simple game with a priori unions (N, W, P), where

$$M^W = \{S_1, S_2, \ldots, S_l\}, \; P = \{P_1, P_2, \ldots, P_u\}, \text{ and } U = \{1, \ldots, u\}.$$

Two trivial partitions of players are given by $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $P^N = \{N\}$. The representatives of a coalition $S \subseteq N$ in the quotient game $(U, \overline{W})$ form a coalition $u(S) \subseteq U$, where $j \in u(S)$ if and only if there is a player $i \in P_j \cap S$. In this way, $u(S)$ is defined as the set of a priori unions involved in the forming of S, that is,

$$u(S) = \{j \in U / P_j \cap S \neq \emptyset\}.$$

Two coalitions S and S' are **equivalent**, if $u(S) = u(S')$, i.e., if their representatives in the quotient game are the same. We will denote by $M^{\overline{W}}$ the **set of MWC in the quotient game**, that is,

$$M^{\overline{W}} = \left\{R \subseteq U / R \in \overline{W} \text{ and } R' \notin \overline{W} \text{ for all } R' \subset R\right\}.$$

Given an a priori union $k \in U$ we will denote by $M_k^{\overline{W}}$ the set of MWC in the quotient game such that the a priori union P_k belongs to each of them, that is,

$$M_k^{\overline{W}} = \left\{R \in M^{\overline{W}} / k \in R\right\}.$$

A coalition $S \in M^W$ is **irrelevant** if $u(S) \notin M^{\overline{W}}$. That is, an irrelevant coalition is a MWC in game (N, W) such that its representatives in the quotient game do not constitute a MWC in $(U, \overline{W})$.

An important assumption of the Owen value is that every coalition $S \subseteq P_k$ has the possibility of forming a winning coalition only joining with one or more of the remaining unions different from P_k, excluding the possibility of forming a coalition by joining S to any proper subsets of some other unions.[1] Taking

[1] In the article by Holler and Nohn (2009) on the PGI with a priori unions this limitation has been dropped.

this limitation into account with respect to the formation of winning coalitions, we introduce the concept of essential subset of a union.

Given a simple game with a priori unions (N, W, P), we will say that a coalition $\emptyset \neq S \subseteq P_k$ is an **essential subset of a union** P_k with respect to R if and only if $R \in M^{\overline{W}}$, $k \in R$, $S \cup \left(\cup_{l \in R \setminus k} P_l\right) \in W$, and $T \cup \left(\cup_{l \in R \setminus k} P_l\right) \notin W$ for every $T \subset S$. $E^{k,R}(N, W, P)$ denotes the set of essential subsets of a union P_k of the game (N, W, P) with respect to R. $E_i^{k,R}(N, W, P)$ denotes the subset of $E^{k,R}(N, W, P)$ formed by coalitions S such that $i \in S$. Finally, $E(N, W, P)$ denotes the set of coalitions S such that there exist a union P_k and $R \in M^{\overline{W}}$ such that $S \in E^{k,R}(N, W, P)$. In order to illustrate the concepts of irrelevant coalition and essential subset of a union, we consider the following example.

EXAMPLE 1 *Take a simple game with a priori unions* (N, W, P) *with* $N = \{a, b, c, d, e, f\}$, $P = \{P_1, P_2, P_3\}$, *where* $P_1 = \{a\}$, $P_2 = \{b, c, d\}$ *and* $P_3 = \{e, f\}$, *and* $M^W = \{S_1, S_2, S_3\}$ *with* $S_1 = \{a, c\}$, $S_2 = \{a, e\}$, *and* $S_3 = \{a, d, f\}$. *Thus,* $U = \{1, 2, 3\}$. *The set of minimal winning coalitions in the quotient game* $\left(U, \overline{W}\right)$ *is*

$$M^{\overline{W}} = \{\{1, 2\}, \{1, 3\}\}.$$

In this game, $S_3 = \{a, d, f\}$ *is a minimal winning coalition. However, its coalition of representatives* $u(S_3) = \{1, 2, 3\}$ *is not minimal in the quotient game. Hence,* S_3 *is irrelevant.*

An essential subset of the union P_2 *with respect to* $R = \{1, 2\}$ *is given by* $\{c\}$ *because* $\{c\} \cup P_1 = \{a, c\} = S_1 \in M^W$.

In the context of simple games with a priori unions, a **coalitional power index** is a function f which assigns an n-dimensional real vector $f(N, W, P)$ to a simple game with a priori unions (N, W, P), where the i-th component of vector $f_i(N, W, P)$ is the power of player i in the game (N, W, P) according to f.

3. The solidarity Public Good Index

In this section, we consider simple games with a priori unions. We assume that a coalitional power index satisfies certain conditions, taking into account the existence of unions. Then, the definition of this variation of the PGI is focused on the quotient game. Firstly, the allocation is divided among the unions and then, it is assumed that the agents inside each union exhibit a solidarity principle. Here, the solidarity principle establishes that players in a priori union have identical power.

We define a new power index called the solidarity Public Good Index.

Definition 1 *Given $(N,W,P) \in SIU(N)$, the solidarity Public Good Index of a player $i \in P_k$ is:*

$$\Theta_i(N,W,P) = \frac{\left|M_k^{\overline{W}}\right|}{\sum_{l \in U}\left|M_l^{\overline{W}}\right|}\frac{1}{|P_k|} = \delta_k\left(U,\overline{W}\right)\frac{1}{|P_k|}. \tag{3}$$

The index Θ is consistent with the previous conditions. Only MWC in the original game that support a MWC in the quotient game have influence. The first term coincides with the Public Good Index of the union P_k in the quotient game. Finally, the term $1/|P_k|$ assures that the payoff for player i is the same as for the other $|P_k|-1$ players of the union P_k. This captures solidarity inside unions. The amount given by this power index to a player $i \in P_k$ is independent of the individual player, i.e., it is the same for all players in P_k. The solidarity Public Good Index coincides with the original PGI if $P = P^n$. However, if $P = P^N$, the solidarity Public Good Index coincides with the egalitarian solution $f_i(N,W,P) = 1/n$ for all $i \in N$ and all $(N,W,P) \in SI(N)$.

In the following example we illustrate the computation of this index.

Example 2 *We consider the simple game with a priori unions given in Example 1. Notice that unions P_2 and P_3 are symmetric in the quotient game. Then, $\delta_1(U,\overline{W}) = 1/2$ and $\delta_2(U,\overline{W}) = \delta_3(U,\overline{W}) = 1/4$. Taking into account the number of players inside each union and the solidarity principle, each player obtains*

$$\Theta_a(N,W,P) = 1/2, \quad \Theta_b(N,W,P) = \Theta_c(N,W,P) = \Theta_d(N,W,P) = 1/12,$$
$$\Theta_e(N,W,P) = \Theta_f(N,W,P) = 1/8.$$

Notice that player b is a null player in the game (N,W). Once the a priori unions and the principle of solidarity are established, player b, according to this power index, has the same power as any other member of his union.

Next, we give definitions of some properties that we will use in the characterization of the solidarity Public Good Index.

Efficiency. A coalitional power index f satisfies efficiency if and only if for every $(N,W,P) \in SIU(N)$, $\sum_{i \in N} f_i(N,W,P) = 1$.

Null union. A coalitional power index f satisfies null union if and only if for every $(N,W,P) \in SIU(N)$ and $k \in U$ such that k is a null player in the quotient game $\left(U,\overline{W}\right)$, $f_i(N,W,P) = 0$ for every $i \in P_k$.

Symmetry among unions. A coalitional power index f satisfies symmetry among unions if and only if for every $(N,W,P) \in SIU(N)$, and $k,l \in U$ symmetric players in the quotient game,

$$\sum_{i \in P_k} f_i(N,W,P) = \sum_{i \in P_l} f_i(N,W,P).$$

These properties are standard in the literature on power indices that take into account a priori unions. The distinguishing property of the coalitional solidarity Public Good Index is the property of solidarity. This property says that players in the same union are awarded in the same way.

Solidarity. A coalitional power index f satisfies solidarity if and only if for every $(N, W, P) \in SIU(N)$, and $i, j \in P_k$, then $f_i(N, W, P) = f_j(N, W, P)$.

The following property is an adaptation of the mergeability property. It is similar to the one proposed in Alonso-Meijide et al. (2009) to characterize the coalitional Deegan-Packel index. We say that two games (N, W, P) and (N, V, P) are mergeable in the quotient game if the corresponding quotient games are mergeable. If two games (N, W, P) and (N, V, P) are mergeable in the quotient game, the mergeability condition guarantees that

$$\sum_{k \in U} \left| M_k^{\overline{W \vee V}} \right| = \sum_{k \in U} \left| M_k^{\overline{W}} \right| + \sum_{k \in U} \left| M_k^{\overline{V}} \right| .$$

The property of PGI-mergeability in the quotient game states that power in a merged game is a weighted mean of power of the two component games, with the sum of the number of MWC of each union in the quotient game of each component game providing the weights.

PGI-mergeability in the quotient game. A coalitional power index f satisfies PGI-mergeability in the quotient game if and only if for any pair $(N, W, P), (N, V, P) \in SIU(N)$ such that $(U, \overline{W})$ and $(U, \overline{V})$ are mergeable, it holds that

$$f(N, W \vee V, P) = \frac{\sum_{k \in U} \left| M_k^{\overline{W}} \right|}{\sum_{k \in U} \left| M_k^{\overline{W \vee V}} \right|} f(N, W, P) + \frac{\sum_{k \in U} \left| M_k^{\overline{V}} \right|}{\sum_{k \in U} \left| M_k^{\overline{W \vee V}} \right|} f(N, V, P) .$$

Independence of superfluous coalitions says that elimination of a minimal winning coalition S of the game such that (a) it is irrelevant or (b) there is a minimal winning coalition $S' \in M^W$ with $u(S) = u(S')$, will not affect the power of the players, since both quotient games have identical set of MWC.

Independence of superfluous coalitions. A coalitional power index f satisfies independence of superfluous coalitions if and only if for any pair $(N, W, P), (N, W', P) \in SIU(N)$ such that $M^{\overline{W}} = M^{\overline{W'}}$ and $M^{W'} = M^W \setminus S$ for some $S \in M^W$, there is $f(N, W, P) = f(N, W', P)$.

THEOREM 3 *The solidarity Public Good Index is the unique coalitional power index defined on $SI(N)$ satisfying the properties of efficiency, null union, symmetry among unions, solidarity, PGI-mergeability in the quotient game, and independence of superfluous coalitions.*

Proof.
Existence. Let (N, W, P) be a simple game with a priori unions where $P = \{P_1, \ldots, P_u\}$ and $U = \{1, \ldots, u\}$. We prove that the solidarity Public Good Index Θ satisfies the properties listed in Theorem 3.

From Theorem 1 and the definition of the solidarity Public Good Index, it is clear that this power index satisfies efficiency, null union, symmetry among unions, and PGI-mergeability in the quotient game. Moreover, it is easy to check that this power index satisfies solidarity. Furthermore, the solidarity Public Good Index satisfies independence of superfluous coalitions since given two simple games (N, W, P), $(N, W', P) \in SIU(N)$ in the conditions of the property, both quotient games have identical set of minimal winning coalitions.

Uniqueness. Let us take a coalitional power index f which satisfies all the above properties. Let us take $(N, W, P) \in SIU(N)$ with $M^W = \{S_1, \ldots, S_l\}$. Since the coalitional power index f satisfies independence of superfluous coalitions, we can assume that $u(S) \in M^{\overline{W}}$ for every $S \in M^W$, and $u(S) \neq u(T)$, for every $S, T \in M^W$.

First, we assume that $l = 1$. In that case, (N, W) is a unanimity game, for instance, (N, W_S) with $S \subseteq N$. Since coalitional power index f satisfies efficiency, null union, symmetry among unions, and solidarity, then f assigns to every player $i \in P_k$

$$f_i(N, W_S, P) = \begin{cases} \frac{1}{|u(S)|}\frac{1}{|P_k|} & \text{if } S \cap P_k \neq \emptyset \\ 0 & \text{if } S \cap P_k = \emptyset \end{cases}.$$

Then, $f_i(N, W_S, P) = \Theta_i(N, W_S, P)$ for every $i \in N$ and for every (N, W_S, P) with $S \subseteq N$.

Let us assume that $l > 1$. Then, $W = W_{S_1} \vee \ldots \vee W_{S_l}$ with $S_j \subset N$ for every $j = 1, \ldots, l$, $M^{\overline{W}} = \{u(S_1), \ldots, u(S_l)\}$, and $u(S_j) \neq u(S_p)$ whenever $j, p = 1, \ldots, l$ $j \neq p$.

Notice that the unanimity games $\left(N, W_{S_j}, P\right)$ and $\left(N, W_{S_p}, P\right)$ for $j, p = 1, \ldots, l$ $(j \neq p)$ are mergeable in the quotient game. Then, by the property of mergeability in the quotient game, it holds that

$$f_i(N, W, P) = \frac{\sum_{j=1}^{l} \sum_{k \in U} \left| M_k^{\overline{W}_{S_j}} \right|}{\sum_{k \in U} \left| M_k^{\overline{W}} \right|} f_i\left(N, W_{S_j}, P\right) = \Theta_i(N, W, P).$$

This finishes the proof. ∎

4. The Owen-extended Public Good Index

In this section, we characterize an extension of the PGI that is similar to Owen's elaborations of the Shapley value (Owen, 1977) and the Banzhaf value (Owen, 1982), and to the extension of the Deegan-Packel index to the case of a priori unions proposed in Alonso-Meijide et al. (2009). We consider two levels of negotiation, (a) among unions, and (b) inside unions. In the process, a player $i \in P_k$ can collaborate with some players $S \subseteq P_k$ and/or with complete unions different from P_k. That is, the potential of a player *joining other unions* is taken into account when we define this index.

Definition 2 *Given* $(N, W, P) \in SIU(N)$*, the Owen-extended Public Good Index of a player* $i \in P_k$ *is:*

$$\Gamma_i(N, W, P) = \frac{1}{\sum_{l \in U} \left| M_l^{\overline{W}} \right|} \sum_{R \in M_k^{\overline{W}}} \frac{\left| E_i^{k,R}(N, W, P) \right|}{\sum_{j \in P_k} \left| E_j^{k,R}(N, W, P) \right|}. \tag{4}$$

The Owen-extended Public Good Index coincides with the original PGI when the a priori unions are given by P^n and P^N.

Next example illustrates the computation of this power index.

Example 3 *We consider the simple game* $(N, W, P) \in SIU(N)$ *defined in Example 1. The set of minimal winning coalitions of the quotient game is* $M^{\overline{W}} = \{\{1, 2\}, \{1, 3\}\}$*. Then, the set of essential subsets for each player with respect to each minimal winning coalition in the quotient game is collected in Table 1.*

Table 1. Essential subsets in Example 1.

Players	Minimal winning coalitions in the quotient game $\{1,2\}$	$\{1,3\}$
a	$\{a\}$	$\{a\}$
b	$\emptyset$	$\emptyset$
c	$\{c\}$	$\emptyset$
d	$\emptyset$	$\emptyset$
e	$\emptyset$	$\{e\}$
f	$\emptyset$	$\emptyset$

Then, the Owen-extended Public Good index assigns to each player the following values

$$\Gamma_a(N, W, P) = 1/2, \quad \Gamma_b(N, W, P) = \Gamma_d(N, W, P) = \Gamma_f(N, W, P) = 0,$$
$$\Gamma_c(N, W, P) = 1/4 = \Gamma_e(N, W, P) = 1/4.$$

Next, we provide an axiomatic characterization of this power index. First, we present the definitions of the well-known properties of null player and symmetry inside unions of a coalitional power index.

Null Player. A coalitional power index f satisfies null player if and only if for every $(N,W,P) \in SIU(N)$ and null player $i \in N$ in the game (N,W), $f_i(N,W,P) = 0$.

Symmetry inside unions. A coalitional power index f satisfies symmetry inside unions if and only if for every $(N,W,P) \in SIU(N)$, and symmetric players $i,j \in P_k$ in the game (N,W), $f_i(N,W,P) = f_j(N,W,P)$.

The following property is another adaptation of the mergeability property. First, we introduce the concept of mergeable games inside unions.

Mergeable games inside unions. Two simple games with a priori unions (N,V,P) and (N,W,P) are mergeable inside unions if:

- (N,W) and (N,V) are mergeable, and
- there is $k \in U$ such that for every $S \in M^W \cup M^V$ there holds $S \subset P_k$.

Notice that in such a case $\{k\}$ is the unique minimal winning coalition in the quotient game. The property of mergeability inside unions states that power in a merged game is a weighted mean of power in each of the two component games, with the sum of the number of MWC for every player in each component game providing the weights.

PGI-mergeability inside unions. A coalitional power index f satisfies mergeability inside unions if for any pair $(N,W,P),(N,V,P)$, of mergeable games inside unions,

$$f(N,W \vee V,P) = \frac{\sum_{j\in N} \left|M_j^W\right|}{\sum_{j\in N} \left|M_j^{W\vee V}\right|} f(N,W,P) + \frac{\sum_{j\in N} \left|M_j^V\right|}{\sum_{j\in N} \left|M_j^{W\vee V}\right|} f(N,V,P).$$

The next two properties are defined in Alonso-Meijide et al. (2009). Invariance with respect to essential subsets of a union says that the power of a player is the same in case we consider two simple games with identical a priori unions, identical sets of essential subsets and, moreover, in both games all MWC have a common set of representatives. Independence of irrelevant coalitions says that elimination of irrelevant coalitions of the game, as defined above, will not change the power of the players. We define the two properties as follows.

Invariance with respect to essential subsets of a union. A coalitional power index f satisfies invariance with respect to essential subsets of a union if and only if for any pair $(N,W,P),(N,V,P)$ such that $E(N,W,P) = E(N,V,P)$, $u(S) = R$, for every $S \in M^W \cup M^V$, $f(N,W,P) = f(N,V,P)$.

Independence of irrelevant coalitions. A coalitional power index f satisfies independence of irrelevant coalitions if and only if for any (N,W,P), and an irrelevant coalition $S \in M^W$, there holds $f(N,W,P) = f(N,W',P)$ where $M^{W'} = M^W \setminus S$.

Theorem 4 *The Owen-extended PGI is the unique coalitional power index defined on $SI(N)$, satisfying the properties of efficiency, null player, symmetry* *inside unions, symmetry among unions, PGI-mergeability in the quotient game, PGI-mergeability inside unions, invariance with respect to essential subsets of a union, and independence of irrelevant coalitions.*

The proof of Theorem 4 is very similar to that of Theorem 1 in Alonso-Meijide et al. (2009) and is therefore omitted.

5. An example

We compute the two coalitional versions of the PGI to analyze the Parliament of Catalonia which resulted from the elections held on November 1st, 2006. This Parliament has also been studied in Carreras et al. (2007). They used binomial semivalues to explain the behavior of one of the parties (ERC).

The Parliament of Catalonia consists of 135 members. Following these elections, the Parliament was composed of:

1. 48 members of CIU, *Convergéncia i Unió*, a Catalan nationalist middle-of-the-road party,
2. 37 members of PSC, *Partido de los Socialistas de Cataluña*, a moderate left-wing socialist party federated to the *Partido Socialista Obrero Español*,
3. 21 members of ERC, *Esquerra Republicana de Cataluña*, a radical Catalan nationalist left-wing party,
4. 14 members of PPC, *Partido Popular de Cataluña*, a conservative party which is a Catalan delegation of the *Partido Popular*,
5. 12 members of ICV, *Iniciativa por Cataluña-Los Verdes-Izquierda Alternativa*, a coalition of ecologist groups and Catalan eurocommunist parties federated to *Izquierda Unida*, and
6. 3 members of $C's$, *Ciudadanos-Partidos de la Ciudadanía*, a non-Catalanist party.

For the sake of clarity, we identify CIU as player 1, PSC as player 2, ERC as player 3, PPC as player 4, ICV as player 5 and $C's$ as player 6. Then, taking $N = \{1,2,3,4,5,6\}$ as the set of players, the corresponding set of minimal winning coalitions is

$$M^W = \{\{1,2\},\{1,3\},\{1,4,5\},\{2,3,4\},\{2,3,5\}\}.$$

We see that $C's$ party is a null player. Two main aspects characterized politics in Catalonia: Spanish centralism versus Catalan autonomy and the ideological dimension of left and right. Taking into account this fact we consider two possible partitions of players:

$$P^1 = \{\{1\},\{2\},\{3,5\},\{4\},\{6\}\} \text{ and}$$

$$P^2 = \{\{1\},\{2,3,5\},\{4\},\{6\}\}.$$

P^1 represents the dimension of Spanish centralism versus Catalanism while P^2 represents the a priori unions that correspond to the left-right dimension. In Table 2, we present the PGI and the two variations of the coalitional PGI for the two partitions of players. The power values indicate that the alternative

Table 2. Some power indices in the Catalonian Parliament as of November 2006

Party	Shares of seats	δ	$\Theta\left(P^1\right)$	$\Theta\left(P^2\right)$	$\Gamma\left(P^1\right)$	$\Gamma\left(P^2\right)$
CIU	0.3556	0.2308	0.3333	0	0.3333	0
PSC	0.2741	0.2308	0.3333	0.3333	0.3333	0.3333
ERC	0.1556	0.2308	0.1667	0.3333	0.25	0.3333
PPC	0.1037	0.1539	0	0	0	0
ICV	0.0889	0.1539	0.1667	0.3333	0.0834	0.3333
C's	0.0222	0	0	0	0	0

interpretation of a priori unions as captured by Θ and Γ matters. Moreover, the focus on Spanish centralism versus Catalan autonomy dimension produces a larger diversity of power than the left-right dimension. Perhaps this is the reason why this dimension is so prominent in the political discussion. Note also that the strongest party, CIU, has no power if the focus is on the left-right dimension, irrespective of whether we apply Θ or Γ. This could be an argument why the dimension of Spanish centralism versus Catalan autonomy is so popular.

6. Final comments

This paper is a part of ongoing research program that analyzes the properties of alternative power measures in order to give substantial characterizations of the measures and prepare for their applications. The underlying perspective is that there is no 'right' or 'wrong' measure: they are indicators, not predictors and as such they might be adequate or inadequate. The authors of this paper share Robert Aumann's view that, in game theory, "different solution concepts are like different indicators of an economy; different methods for calculating a price index; different maps (road, topo, political, geologic, etc., not to speak of scale, projection, etc.); different stock indices (Dow Jones, ...). They depict or illuminate the situation from different angles; each one stresses certain aspects at the expense of others" (Aumann, 1977, p.464). However, to interpret the indicators and to apply them adequately, one has to know their properties. This, of course, is a major task, given the multitude of power measures, so far developed, and the large variation in the situations to which these measures are, or should be, applied. Moreover, this program risks, like all successful research programs, the fate that no foreseeable end exists.

This paper extended the Public Good Index as defined in Holler (1982) to simple games with a priori unions. Due to its normalization, this index measures relative power, only. There is no probability interpretation for this measure that can be used to express the (absolute) power of an individual player to change the decision of the voting body under the given rule. The relative measure expresses how much stronger player i is compared to player j in determining the outcome, but both players could have hardly any influence at all if the forming of a winning coalition is difficult, e.g., because of a large majority quota. Under the label of Public Value, Holler and Li (1995) discuss a non-normalized form of the Public Good Index, measuring absolute power, and extend it to general games. The Public Value of player i is identical with the number of minimum winning coalitions that have i as a member, i.e., $|M_i|$. Turnovec (2010) also assumes a non-normalized form of the Public Good Index (in fact, a linear transformation of Public Value) and offers a decomposition into two factors: factor one gives the probability of player i being a member of a minimal winning coalition and factor two expresses the probability that a minimal winning coalition will be formed, given that all coalitions form with equal probability. Related to this approach, Brueckner (2002) presents two variations of the Public Good Index that allow for a probabilistic interpretation of power in the case of homogeneous and partially homogeneous voting behavior.

To our knowledge, none of these approaches has been applied to develop extensions of the Public Good Index with a priori unions so far, although it could be interesting to see the impact of a priori unions on the probability of an individual player to determine the outcome of a coalition game, and what properties the measures have that can be applied to answer this question. Perhaps there will be answers to these questions in the near future.

In this paper we did not provide a method to easily compute the two versions of the Public Good Index with a priori unions considered. The multilinear extension (Owen, 1972) has been used to compute the Owen coalition value (Owen and Winter, 1992), the Banzhaf-Owen coalition value (Carreras and Magaña, 1994), and the symmetric coalition Banzhaf value (Alonso-Meijide et al., 2005). In the particular case of weighted majority games with a priori unions, procedures based on generating functions have been described in Alonso-Meijide and Bowles (2005) to compute power indices. These methodologies can be applied to compute the coalitional power indices analyzed in the current paper.

Acknowledgements

The authors wish to thank Andreas Nohn and two anonymous referees for interesting suggestions and comments. This research received financial support from Spanish Ministry for Science and Technology and FEDER through Project ECO2008-03484-C02-02 and from Xunta de Galicia through Project INCITE09–207–064–PR.

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